The relation between uniform stability and comparison functions

Consider the system : $\dot{x} = f(x,t)$, $f(0,t) = 0$, $\forall t \geq t_0$

Lemma 4.5 :

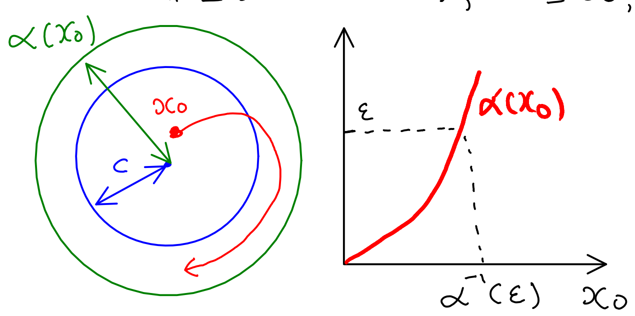
- 0 is uniformly stable if and only if there exist a class K function α and $c > 0$, independent of t_0, such that :
$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \forall t \geq t_0, \forall \|x(t_0)\| < c$$

- 0 is uniformly asymptotically stable if and only if there exist a class KL function β and $c > 0$ independent of t_0, such that:
$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \forall t \geq t_0, \forall \|x(t_0)\| < c$$

- 0 is globally uniformly asymp. stable if and only if there exists a class KL function β independent of t_0, such that
$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \forall t \geq t_0$$

Proof for uniform stability :

(If) Suppose that there exist a class K function α and $c > 0$ such that
$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \forall t \geq t_0, \forall \|x(t_0)\| < c$$

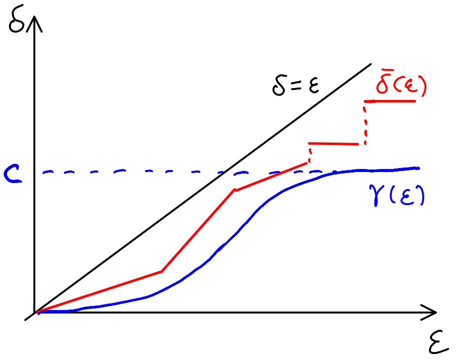


For any given $\varepsilon > 0$, take $\delta < \min(c, \alpha^{-1}(\varepsilon))$

(only if) Suppose that for every $\varepsilon > 0$, $\exists\, \delta(\varepsilon)$ such that
$$\|x(t_0)\| < \delta(\varepsilon) \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0$$

$\delta(\varepsilon)$ is not unique but has supremum. Define $\bar{\delta}(\varepsilon) = \sup \delta(\varepsilon)$

Note that $\bar{\delta}(\varepsilon)$ is nondecreasing and positive definite,



We can always construct a class K function γ such that $\gamma(\varepsilon) \le b \cdot \bar{\delta}(\varepsilon)$, $b < 1$

Define $\alpha(\delta) \triangleq \gamma^{-1}(\delta) \rightarrow$ class $\mathcal{K}$

$c \triangleq \sup_{\varepsilon} \gamma(\varepsilon)$

Thus $\forall \|x(t_0)\| < c$, $\forall t \ge t_0$,

$\|x(t)\| \le \alpha(\|x(t_0)\|)$

Lyapunov Theory for TV systems

Thm 4.8 (modified) Let $V(x,t): D \times [0,\infty) \rightarrow \mathbb{R}$ be a continuously differentiable function such that:

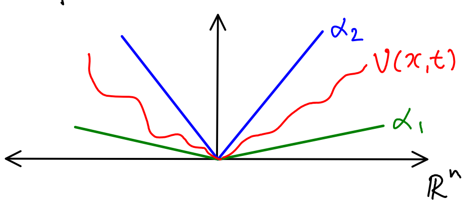
$$\alpha_1(\|x\|) \le V(x,t) \le \alpha_2(\|x_2\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \le 0$$

for all $t \ge 0$ and $x \in D$, where α_1 and α_2 are class k functions

Then 0 is uniformly stable.

Proof: Notice that $\frac{dV}{dt} \le 0$, thus $V(x,t)$ is nonincreasing in t.



For any $\varepsilon > 0$, pick ε' such that $B(0,\varepsilon') \in D$

Pick $\delta \le \alpha_2^{-1}(\alpha_1(\varepsilon'))$

$$\|x(t_0) < \delta\| \Rightarrow \alpha_2(\|x(t_0)\|) < \alpha_1(\varepsilon')$$

$$V(x(t_0), t_0) < \alpha_1(\varepsilon')$$

$$V(x(t), t) < \alpha_1(\varepsilon')$$

$$\alpha_1(\|x(t)\|) < \alpha_1(\varepsilon') \Rightarrow \|x(t)\| < \varepsilon'$$

Thm 4.8 is more general, since instead of α_1 and α_2, the bounds are given by $W_1(x)$ and $W_2(x)$ that are continuous and +def functions in D. However, we can use the "sandwich theorem" and have $\alpha_1(\|x\|) \leq W_1(x)$ and $W_2(x) \leq \alpha_2(\|x\|)$.

Thm 4.9 (modified): Take the previous theorem, and suppose that instead of $\frac{dV}{dt} \leq 0$, we have that $\frac{dV}{dt} \leq -\alpha_3(\|x\|)$, where α_3 is a class K function then 0 is uniformly asymp. stable

Proof: Pick r such that $B(0,r) \subset D$, then pick $c < \alpha_2^{-1}(\alpha_1(r))$

For any $\|x(t_0)\| \leq c$, $x(t) \in D$ $\forall t \geq t_0$, thus the following holds

$$\dot{V} \leq -\alpha_3(\|x\|)$$
$$\leq -\alpha_3(\alpha_2^{-1}(V))$$
$$\leq -\alpha(V)$$, where α is a class K function (Lipschitz)

Let $y(t)$ satisfy $\dot{y} = -\alpha(y)$, $y(t_0) = V(x(t_0), t_0)$

$y(t) = \sigma(V(x(t_0)), t - t_0)$ where σ is a class KL function

$V(t) \leq \sigma(V(x(t_0), t_0), t - t_0) \leq \sigma(\alpha_2(\|x(t_0)\|), t - t_0)$

Thus: $\alpha_1(\|x(t)\|) \leq \sigma(\alpha_2(\|x(t_0)\|), t - t_0)$

$$\|x(t)\| \leq \underbrace{\alpha_1^{-1}(\sigma(\alpha_2(\|x(t_0)\|), t - t_0))}_{\text{class KL function}}$$

Addendum: If $D = \mathbb{R}^n$ and α_1, α_2 are K_∞ then 0 is globally uniformly asymp. stable.

Example:
$$\dot{x}_1 = -x_1 - g(t)x_2$$
$$\dot{x}_2 = x_1 - x_2$$

where $g(t)$ is continuously differentiable and satisfies

$0 \leq g(t) \leq k$ and $\dot{g}(t) \leq g(t)$, $\forall t \geq 0$

Pick $V(x,t) = x_1^2 + (1+g(t))x_2^2$, thus:

$$\underbrace{x_1^2 + x_2^2}_{\substack{\alpha_1(\|x\|) \\ K\infty}} \leq V(x,t) \leq x_1^2 + (1+k)x_2^2 \leq \underbrace{(1+k)(x_1^2+x_2^2)}_{\substack{\alpha_2(\|x\|) \\ K\infty}}$$

$$\frac{dV}{dt} = x_2^2 \dot{g}(t) + 2x_1(-x_1 - g(t)x_2) + 2(1+g(t))x_2(x_1 - x_2)$$
$$= x_2^2 \dot{g}(t) - 2x_1^2 - \cancel{2x_1x_2 g(t)} + 2x_1x_2 - 2x_2^2 + \cancel{2x_1x_2 g(t)} - 2g(t)x_2^2$$

$$= -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

but: $2 + 2g(t) - \dot{g}(t) \geq 2 + g(t) \geq 2$

Thus $\frac{dV}{dt} \leq -2x_1^2 + 2x_1x_2 - 2x_2^2$

$$\leq \underbrace{-(x_1+x_2)^2 - x_1^2 - x_2^2}_{-W_3(x),\ \text{continuous},\ +\text{def}}$$

Thus 0 is globally uniformly stable.

Example: $\dot{x} = A(t)x$, $A(t)$ continuous

Define $V(x,t) = x^T P(t) x$, where $P(t)$ is continuously differentiable and

$$\underbrace{c_1 x^T x}_{\alpha_1(\|x\|),\ K\infty} \leq x^T P(t) x \leq \underbrace{c_2 x^T x}_{\alpha_2(\|x\|),\ K\infty} , \quad c_1, c_2 > 0$$

$$\frac{dV}{dt} = x^T \dot{P}(t) x + x^T A^T(t) P(t) x + x^T P(t) A(t) x$$
$$= x^T \underbrace{(\dot{P}(t) + A^T(t)P(t) + P(t)A(t))}_{\triangleq Q(t)} x$$

If $Q(t) \leq -c_3 I$ for some $c_3 > 0$, then 0 is globally uniformly asymp. stable