Last lecture:

$G(s) \to G(j\omega)$

PR, SPR of a TF $G(s)$

Example: $\frac{1}{s}$ PR (not SPR) $\frac{1}{s+1}$: SPR

Freq. Concept		Time domain
$G(s)$ PR	$\xrightarrow{\text{PR Lemma}}$	(A, B, C, D) passive
$G(s)$, SPR	$\xrightarrow{\text{KYP Lemma}}$	(A, B, C, D) strictly passive

Remark

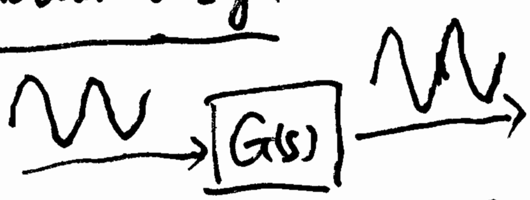
- $G(s)$ is PR $\to$ the relative degree of $G(s) \le 1$

  e.g. $\boxed{\frac{1}{s^2+s+1}}$

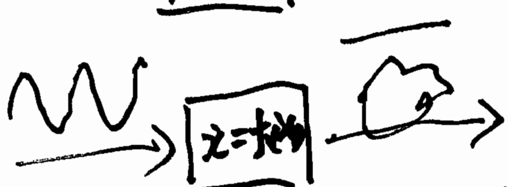
- $\frac{s+c}{s^2+bs+a}$: PR $\longrightarrow 0 \le c \le b$, $a, b > 0$

  SPR $\longrightarrow 0 < c < b$, $a, b > 0$

Frequency Domain Design



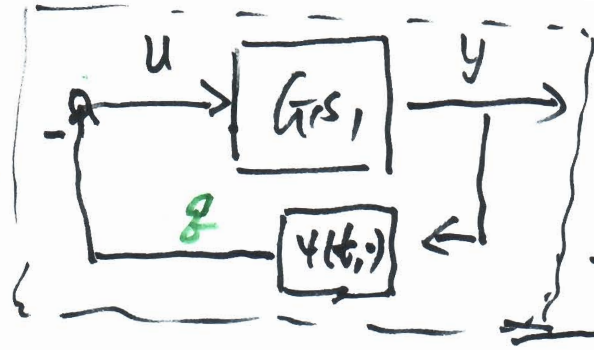
Bode
Nyquist

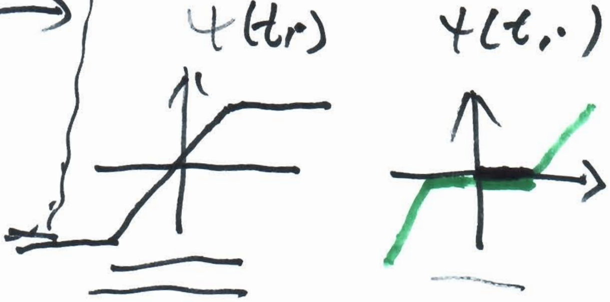


Nyquist [$G(j\omega)$]

For special cases, use linear frequency concepts to understand a class of nonlinear systems!

(*)

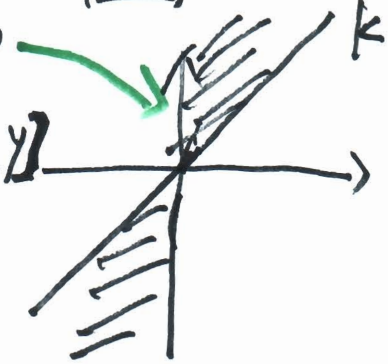


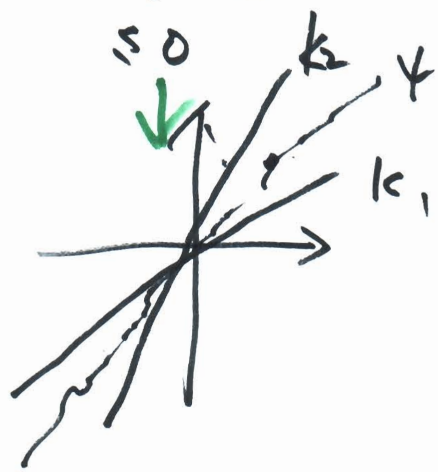


$\psi \in [0,1]$ $\qquad$ $\psi \in [0,1]$

- $\psi(t,\cdot)$: sector nonlinearity

- $\psi(t,\cdot) \in [0,\infty]$ $\quad y^T \psi(t,y) \geq 0$
- $\psi(t,\cdot) \in [k,\infty]$ $\quad y^T(\psi(t,y) - ky) \geq 0$
- $\psi(t,\cdot) \in [k_1,k_2]$ $\quad [\psi(t,y) - k_1 y]^T [\psi(t,y) - k_2 y] \leq 0$





- Absolute stability of (*):

$$\underline{G(s)}: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad u = -\underline{\underline{\psi}}(t,y)$$

For a given sector nonlinearity ψ, the origin of x is globally uniformly asymptotically stable.

- Two THMs: Circle Criterion ; Popov Criterion

Circle Criterion

I. $\psi \in [0, \infty]$

The condition: $G(s)$ is SPR, s.t. the $(*)$ is absolutely stable.

Proof: $G(s)$ is SPR $\xrightarrow{\text{KYP Lemma}}$ $\exists P,\ P = P^T > 0$. s.t.

$V = \frac{1}{2} x^T P x$ is a valid storage function for $G(s)$.

$$\dot{V} = -\frac{1}{2}\varepsilon x^T W x + u^T y, \quad W = W^T > 0$$

$G(s)$: $\dot{x} = Ax + Bu$, $y = Cx + Du$ $\quad\Downarrow$ $G(s)$ is strictly passive

$$u = -\psi(t, y)$$

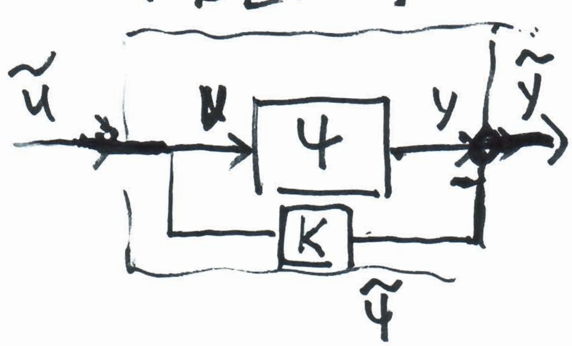
$$\dot{V} = -\frac{1}{2}\varepsilon x^T W x + (-\psi^T y)$$

Because $\psi \in [0, \infty]$, $-\psi^T y \leq 0$, $\dot{V} \leq -\frac{1}{2}\varepsilon x^T W x$

$\Rightarrow$ $x = 0$ is g.u.a.s.

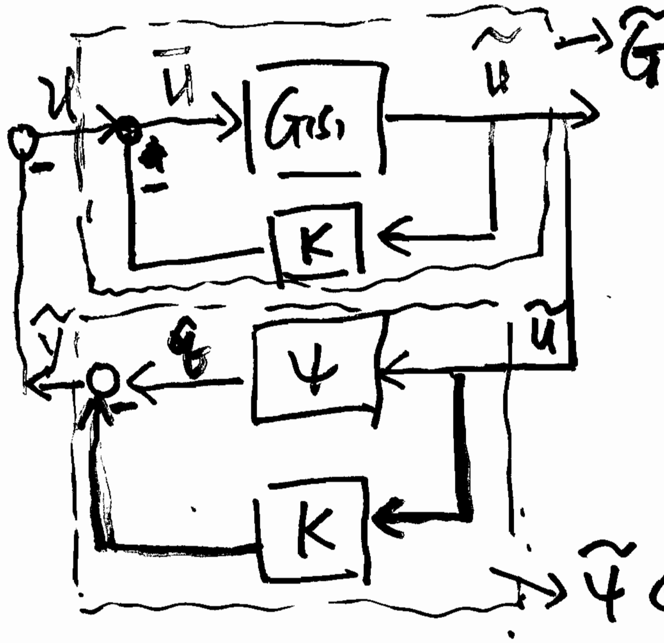
II. $\psi \in [K, \infty]$

Transform ψ to $\tilde{\psi} \in [0, \infty]$, back to Case I.



$$u^T(\underbrace{y - Ku}_{=\tilde{y}}) \geq 0, \quad u = \tilde{u}$$

$$\tilde{u}^T \tilde{y} \geq 0$$



$\tilde{y} = q - K\hat{u}$

$u = K\hat{u} - q$

$\bar{u} = u - K\hat{u} = -q$

From Case I, we know that if $\tilde{G}(s)$ is SPR, then we have absolute stability.

$\tilde{G}(s) = \dfrac{G(s)}{1 + KG(s)}$, when K is a scalar

$\tilde{G}(s) = G(s)[I + KG(s)]^{-1}$, K is a matrix.

$\boxed{\text{The condition: } \tilde{G}(s) \text{ is SPR.}}$

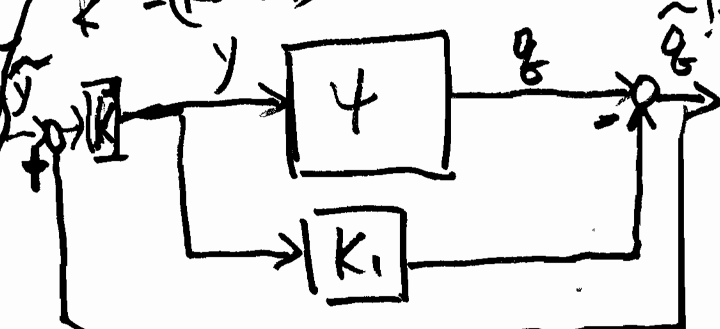
Case III: $\psi \in [K_1, K_2] \longrightarrow \tilde{\psi} \in [0, \infty]$

$(q - K_1 y)^T (q - K_2 y) \le 0$

$\tilde{q} \qquad -\tilde{y}$

$\tilde{\psi}$: input $\tilde{y}$, output $\tilde{q}$

$K^{-1} = (K_2 - K_1)^{-1}$



$\tilde{y}^T \tilde{q} \ge 0$

$\Downarrow$

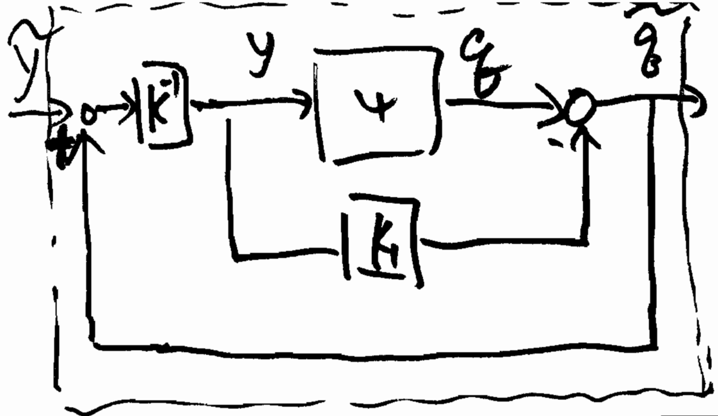
$\tilde{\psi} \in [0, \infty]$

$\tilde{q} = q - K_1 y$

$-\tilde{y} = q - K_2 y$

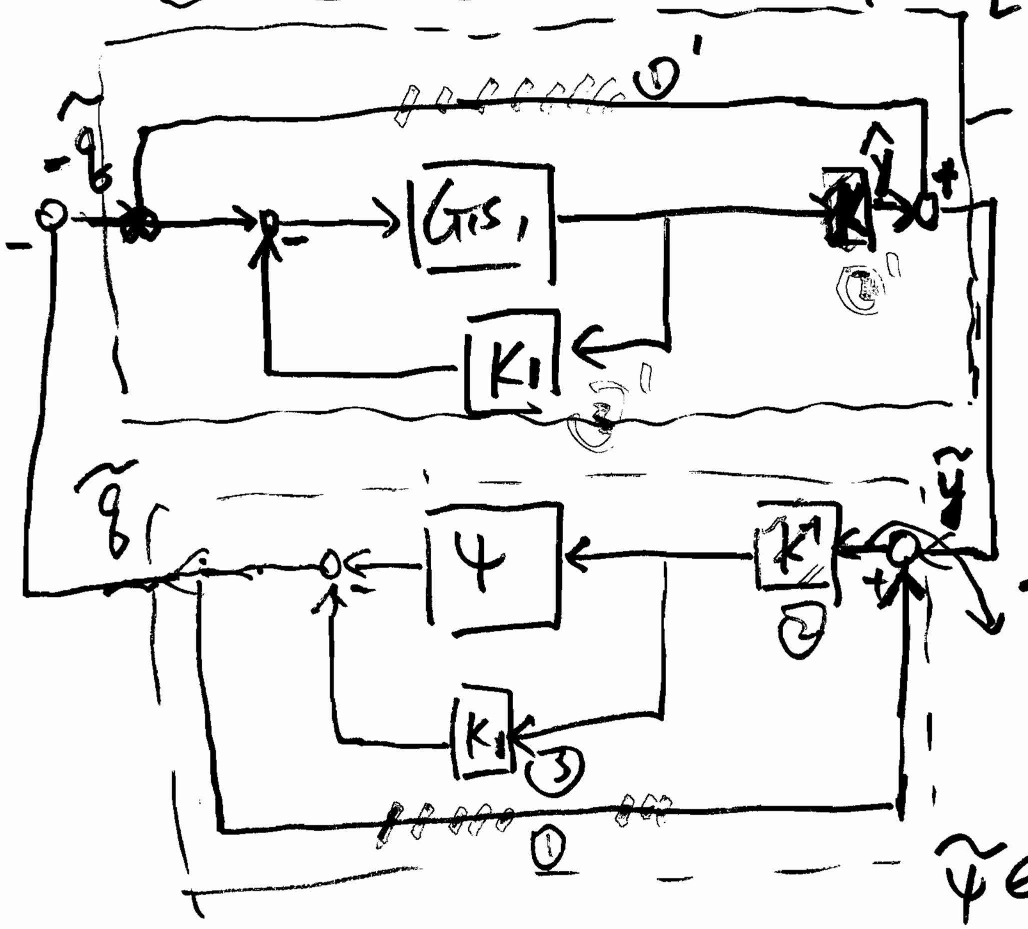
$\tilde{y} = K_2 y - q$

$y = (K_2 - K_1)^{-1}(\tilde{y} + \tilde{q}) \Leftarrow \tilde{y} = -\tilde{q} + (K_2 - K_1) y \Leftarrow = \underbrace{K_1 y - q}_{-\tilde{q}} + (K_2 - K_1) y$



$K = K_2 - K_1$

$\tilde{\psi} \in [0, \infty]$



$\tilde{G}(s)$

$\tilde{y} = \hat{y} + (-\tilde{q})$

$\bar{y} = \hat{y} + \tilde{q}$

$\bar{y} = \hat{y}$

$\tilde{\psi} \in [0, \infty]$

<u>From Case I</u>: $\tilde{G}(s)$ is SPR $\Rightarrow$ Absolute stability

K is scalar: $\tilde{G}(s) = 1 + K \frac{G(s)}{1 + K_1 G(s)} = \frac{1 + K_2 G(s)}{1 + K_1 G(s)}$

$$\tilde{G}(s) = [I + K_2 G(s)][I + K_1 G(s)]^{-1}$$

Summary (THM 7.1) (*) is absolutely stable if.

- $\psi \in [0,\infty]$, $G(s)$ is SPR
- $\psi \in [K,\infty]$, $G(s)(I+KG(s))^{-1}$ is SPR
- $\psi \in [K_1,K_2]$, $[I+K_2G(s)][I+K_1G(s)]^{-1}$ is SPR

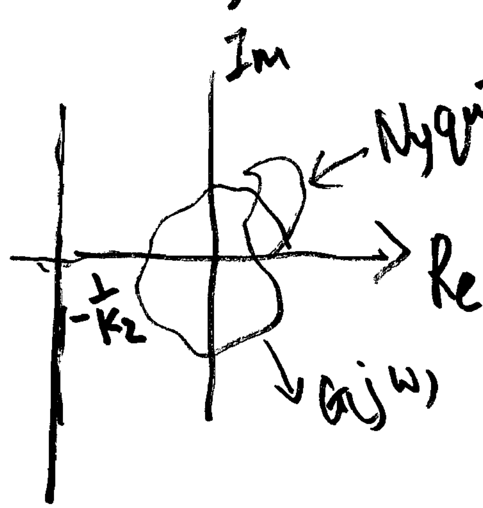
Remark

- ~~THM~~ THM 7.1 called Multivariable $\boxed{\text{Circle}}$ criterion
- Scaler cases: $\psi \in [K_1,K_2]$, K_1, K_2 scalars, $K_2 > K_1$

$$Z(s) = \frac{1+K_2G(s)}{1+K_1G(s)} \text{ SPR } \xrightarrow{\text{Lemma 6.1}} \begin{cases} Z(s) \text{ Hurwitz} \\ \mathrm{Re}[Z(j\omega)] > 0 \ \forall \omega. \end{cases}$$

① $K_1 = 0$

$Z(s) = 1+K_2G(s)$, $\quad \mathrm{Re}[Z(j\omega)] = \mathrm{Re}[1+K_2\underline{G(j\omega)}] > 0$

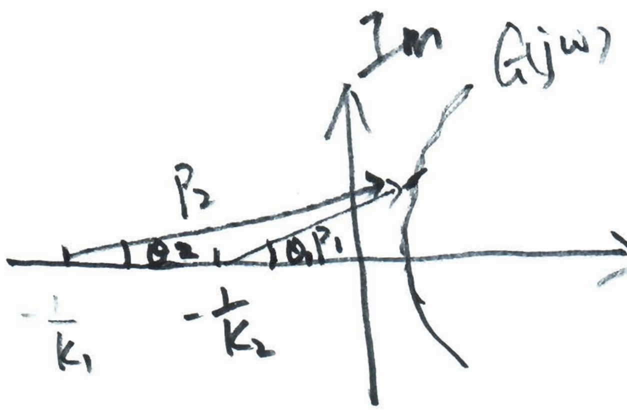


$$\Downarrow$$

$$\begin{cases} K_2 \mathrm{Re}[\frac{1}{K_2} + G(j\omega)] > \\ K_2 > 0 \\ \mathrm{Re}[\frac{1}{K_2} + G(j\omega)] > \end{cases}$$

② K_1, K_2 are of the same sign, $K_1 K_2 > 0$

$$\mathrm{Re}\left[\frac{1+K_2 G(j\omega)}{1+K_1 G(j\omega)}\right] > 0 \Rightarrow \cancel{\frac{K_2}{K_1}}\ \boxed{\mathrm{Re}\left[\frac{\frac{1}{K_2}+G(j\omega)}{\frac{1}{K_1}+G(j\omega)}\right] > 0}$$



$$P_1 = \frac{1}{K_2} + G(j\omega) = G(j\omega) - \left(-\frac{1}{K_2}\right)$$

$$P_2 = \frac{1}{K_1} + G(j\omega) = G(j\omega) - \left(-\frac{1}{K_1}\right)$$

$$\mathrm{Re}\left[\frac{P_1}{P_2}\right] > 0$$

$$P_1 = |P_1| e^{j\theta_1}$$

$$P_2 = |P_2| e^{j\theta_2}$$

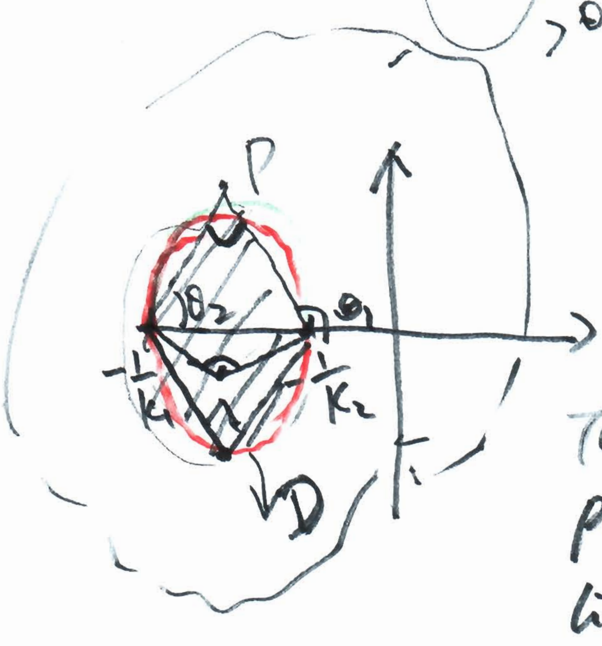
$$\mathrm{Re}\left[\frac{|P_1|}{|P_2|} e^{j(\theta_1-\theta_2)}\right] > 0 \Rightarrow \cos(\theta_1-\theta_2) > 0$$

$\frac{|P_1|}{|P_2|} > 0$

$$\Downarrow$$

$$\boxed{|\theta_1 - \theta_2| < \frac{\pi}{2}}$$

$$|\theta_1 - \theta_2| = \frac{\pi}{2}$$

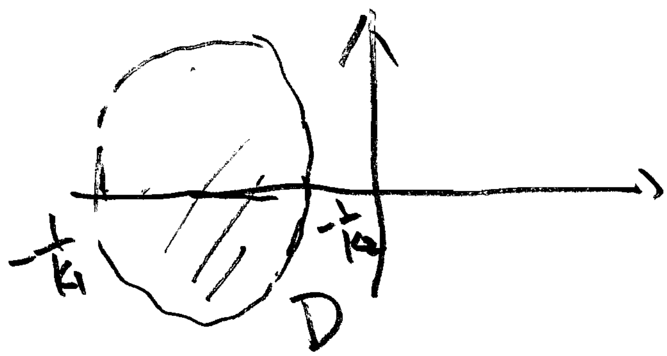


The nyquist plot has to lie outside the circle D.

P

③ K_1, K_2 of different signs.

$$\mathrm{Re}\left[\frac{\frac{1}{K_2}+G(j\omega)}{\frac{1}{K_1}+G(j\omega)}\right]<0$$



The Nyquist plot of $G(j\omega)$ has to lie inside the circle of D.