Stability of Perturbed Systems

Consider a system: $\dot{x} = \underbrace{f(x,t)}_{\text{nominal system}} + \underbrace{g(x,t)}_{\text{perturbation}}$

$f(0,t) = 0, \ \forall t \geq 0$, $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are piecewise continuous in t and (locally) Lipschitz in x

Question: If the nominal system is stable (in some sense), what can be said about the stability of the perturbed system?

$\boxed{\text{Vanishing perturbation: } g(0,t) = 0, \ \forall t \geq 0}$

* Suppose that the origin is exponentially stable for the nominal system with $V(x,t)$ continuously differentiable and positive definite such that:

$$c_1\|x\|^2 \leq V(x,t) \leq c_2\|x\|^2$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -c_3\|x\|^2$$
$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x\|$$

for $c_1, c_2, c_3, c_4 > 0$.

a) Suppose that the perturbation satisfies: $\|g(x,t)\| \leq \gamma\|x\|$, $\gamma > 0$, then

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) + \frac{\partial V}{\partial x} g(x,t) \leq -c_3\|x\|^2 + c_4\gamma\|x\|^2$$
$$\leq (c_4\gamma - c_3)\|x\|^2$$

Thus, if $\gamma < \frac{c_3}{c_4}$, the system is still exponentially stable

This holds for both local and global exponential stability

b) Higher order perturbation : $\|g(x,t)\| \le \gamma \|x\|^{1+\varepsilon}$, $\varepsilon > 0$, $\gamma > 0$

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) + \frac{\partial V}{\partial x} g(x,t) \le -C_3\|x\|^2 + C_4\gamma\|x\|^{2+\varepsilon}$$
$$= -\frac{C_3}{2}\|x\|^2 + \left(-\frac{C_3}{2}\|x\|^2 + C_4\gamma\|x\|^{2+\varepsilon}\right)$$
$$\le -\frac{C_3}{2}\|x\|^2 \quad \text{if} \quad \|x\| \le \left(\frac{C_3}{2C_4\gamma}\right)^{1/\varepsilon}$$

Local exponential stability is preserved

* Suppose that the nominal system is only uniformly asymptotically stable, with Lyapunov function $V(x,t)$:

$W_1(x) \le V(x,t) \le W_2(x)$, W_1, W_2 are continuous + def functions

$$\left.\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) &\le -C_3\,\phi^2(x) \\ \left\|\frac{\partial V}{\partial x}\right\| &\le C_4\,\phi(x) \end{aligned}\right\} \text{quadratic type Lyapunov function}$$

where $C_3, C_4 > 0$, $\phi(\cdot)$ is a continuous + def function.

If $\|g(x,t)\| \le \gamma \cdot \phi(x)$, then if $\gamma \le \frac{C_3}{C_4}$, the uniform asymp. stability is retained

Example: $\dot{x} = \underbrace{-x^3}_{\text{nominal}} + g(x,t)$

Pick $V(x) = x^4 \rightarrow \frac{dV}{dt} = 4x^3 \cdot -x^3 = -4x^6$

$$\left\|\frac{dV}{dx}\right\| = 4\|x\|^3$$

Thus, pick : $\phi(x) = \|x\|^3$, $C_3 = C_4 = 4$: If $\|g(x,t)\| \le \gamma\|x\|^3$, with $\gamma < 1$ the perturbed system is still uniformly asymp. stable

$\boxed{\text{Nonvanishing Perturbation : } g(0,t) \neq 0}$

If $\|g(x,t)\| \leq \delta$: consider $g(x,t)$ as input, and apply ISS theory

Suppose that the origin is exponentially stable for the nominal system with $V(x,t)$ continuously differentiable and positive definite such that:

$$c_1\|x\|^2 \leq V(x,t) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -c_3\|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x\|$$

for $c_1, c_2, c_3, c_4 > 0$.

If $\|g(x,t)\| \leq \delta \Rightarrow \frac{dV}{dt} \leq -c_3\|x\|^2 + c_4\|x\| \cdot \delta$

$$= -\frac{c_3}{N}\|x\|^2 + \left(-\frac{c_3(N-1)}{N}\|x\|^2 + c_4\|x\|\delta \right)$$

$$\leq -\frac{c_3}{N}\|x\|^2 \quad \text{if} \quad \|x\| \geq \frac{N}{N-1}\frac{c_4}{c_3}\delta$$

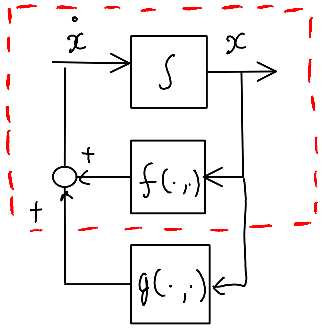
$\alpha_1(\|x\|) \triangleq c_1\|x\|^2$, $\alpha_2(\|x\|) \triangleq c_2\|x\|^2$, $\rho(\delta) \triangleq \frac{N}{N-1}\frac{c_4}{c_3}$

$x(t) \leq \beta(\|x(0)\|, t) + \gamma \cdot \delta$, with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

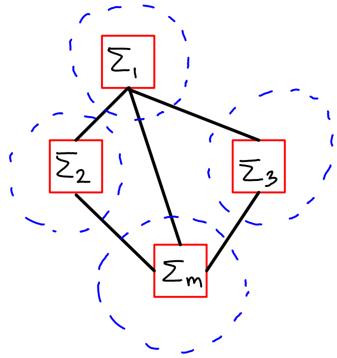
$\gamma > \sqrt{\frac{c_2}{c_1}} \cdot \frac{c_4}{c_3}$

exponential convergence
see Exercise 4.51

Interconnected Systems



perturbation analysis



Interconnected system : view the other sub-systems as "perturbation"

$$\dot{x}_i = f_i(x_i, t) + g_i(x, t) \quad , i = 1 \dots m \quad , x_i \in \mathbb{R}^{n_i}$$

$$f_i(0,t) = 0 \ , \ g_i(0,t) = 0 \ , \ \forall t \geq 0$$

Assume f_i and g_i are smooth enough for existence and uniqueness

Suppose that each Σ_i is uniformly asymp. stable, such that we can design quadratic type Lyapunov functions $V_i(x,t)$:

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(x_i, t) \leq -\alpha_i \phi_i^2(x)$$

$$\left\| \frac{\partial V_i}{\partial x_i} \right\| \leq \beta_i \phi_i(x) \quad \text{, where } \alpha_i, \beta_i > 0 \text{ and } \phi_i(\cdot) \text{ is continuous + def}$$

Suppose that the interconnection is weak enough such that :

$$\| g_i(x,t) \| \leq \sum_{j=1}^{m} \gamma_{ij} \phi_j(x_j)$$

Then : Propose $v(x) = \sum_{i=1}^{m} d_i V_i(x_i, t)$, we have

$$\frac{dV}{dt} = \sum_{i=1}^{m} d_i \frac{\partial V_i}{\partial t} + \sum_{i=1}^{m} d_i \frac{\partial V_i}{\partial x_i} \left(f_i(x_i, t) + g_i(x,t) \right)$$

$$\frac{dV}{dt} = \sum_{i=1}^{m} d_i \left[\left(\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(x_i,t) \right) + \frac{\partial V_i}{\partial x_i} g_i(x,t) \right]$$

$$\leq \sum_{i=1}^{m} d_i \left(-\alpha_i \phi_i^2(x_i) + \beta_i \phi_i(x_i) \sum_{j=1}^{m} \gamma_{ij} \phi_j(x_j) \right)$$

$$\leq - \begin{pmatrix} \phi_1(x_1) \\ \vdots \\ \phi_m(x_m) \end{pmatrix}^T \left[\begin{pmatrix} d_1\alpha_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & d_m\alpha_m \end{pmatrix} - \begin{pmatrix} d_1\beta_1\gamma_{11} & d_2\beta_2\gamma_{12} & \cdots \\ \vdots & \ddots & \\ d_m\beta_m\gamma_{m1} & \cdots & d_m\beta_m\gamma_{mm} \end{pmatrix} \right] \begin{pmatrix} \phi_1(x_1) \\ \vdots \\ \phi_m(x_m) \end{pmatrix}$$

Define: $S = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_m \end{pmatrix} - \begin{pmatrix} \beta_1\gamma_{11} & \beta_1\gamma_{12} & \cdots & \beta_1\gamma_{1m} \\ \vdots & & & \vdots \\ \beta_m\gamma_{m1} & \cdots & \cdots & \beta_m\gamma_{mm} \end{pmatrix}$

$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix}$

$$\frac{dV}{dt} \leq \frac{-1}{2} \begin{pmatrix} \phi_1(x_1) \\ \vdots \\ \phi_m(x_m) \end{pmatrix}^T \left(DS + S^T D \right) \begin{pmatrix} \phi_1(x_1) \\ \vdots \\ \phi_m(x_m) \end{pmatrix}$$

$\boxed{\text{Goal: Design } D \text{ such that } (DS + S^T D) > 0}$

Lemma: such D exists if and only if S is an M-matrix, that is

$$\det \begin{pmatrix} S_{11} & \cdots & S_{1K} \\ \vdots & & \vdots \\ S_{K1} & \cdots & S_{KK} \end{pmatrix} > 0 \quad , \text{for } K = 1, 2, \ldots m$$

Interpretation: the coupling between subsystems should be "weak" enough

Special case, $m=2$: $S_{11} = \alpha_1 - \beta_1\gamma_{11} > 0$

$$\det \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} > 0 \to (\alpha_1 - \beta_1\gamma_{11})(\alpha_2 - \beta_2\gamma_{22}) - \underbrace{\beta_1\beta_2\gamma_{12}\gamma_{11}}_{\text{Coupling between } \Sigma_1 \text{ and } \Sigma_2} > 0$$