Example:

$$\begin{cases} \dot{x}_1 = -a x_1 + u \checkmark \\ \dot{x}_2 = -b x_2 + K - c x_1 x_3 \\ \dot{x}_3 = \theta x_1 x_2 \checkmark \end{cases} \qquad a, b, c, K, \theta > 0$$

$$y = x_3$$

$$\dot{y} = \dot{x}_3 = \theta x_1 x_2 \qquad r > 1$$

$$\ddot{y} = \theta \dot{x}_1 x_2 + \theta x_1 \dot{x}_2$$

$$= \theta x_2(-a x_1 + u) + \theta x_1(-b x_2 + K - c x_1 x_3)$$

$$= \theta x_2 u + \alpha(x)$$

When $x_2 \neq 0$, $r = 2$.

In $D = \{(x_1, x_2, x_3) \mid x_2 \neq 0\}$, $r = 2$.

$$\left.\begin{array}{l} \ddot{y} = \theta x_2 u + \alpha(x) \\ u = \frac{1}{\theta x_2}(-\alpha(x) + v) \end{array}\right\} \Rightarrow \ddot{y} = v$$

<u>zero dynamics</u>: $y \equiv 0$, $v = 0$

$$y \equiv 0 \Leftrightarrow x_3 \equiv 0 \Rightarrow \dot{x}_3 = 0 \Rightarrow \theta x_1 x_2 = 0$$

$$\theta \neq 0, \; x_2 \neq 0 \;\Rightarrow\; x_1 = 0$$

$y \equiv 0,\ x_1 = 0 \Rightarrow \dot{x}_2 = -b x_2 + k, \quad b > 0.$

equilibrium: $x_2 = \frac{k}{b}$

The zero dynamics is a.s. $\Rightarrow$ minimum phase

$\dot{x} = f(x) + g(x) u \xrightarrow{z = T(x)} \dot{z} = Az + B[\alpha(x) + \gamma(x) u]$

$x \in R^n$

(A, B): controllable

$\gamma(x)$: nonsingular

When is the system feedback linearizable?

Result 1: A guided search for $z = T(x)$

$\dot{x} = f(x) + g(x) u$ is feedback linearizable

iff $\exists$ a function $h(x)$, s.t.

the rel. deg. from u to $y = h(x)$ is

$r = n$!

Proof: (only sufficiency).

Define:
$$\begin{cases} z_1 = h(x) \\ z_2 = \dot{z}_1 = \frac{\partial h}{\partial x} f(x) = L_f h(x) \\ z_3 = \dot{z}_2 = L_f^2 h(x) \\ \vdots \\ z_n = \dot{z}_{n-1} = L_f^{n-1} h(x) \end{cases} \qquad z = T(x)$$

$$\Rightarrow \quad \begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= \underbrace{L_f^n h(x)}_{\alpha(x)} + \underbrace{L_g L_f^{n-1} h(x)}_{\neq 0 \text{ invertible}} u \end{aligned}$$

$$u = \frac{1}{L_g L_f^{n-1} h(x)} \left[-L_f^n h(x) + v \right]$$

$$\Rightarrow \begin{cases} \dot{z}_1 = z_2 \\ \vdots \\ \dot{z}_n = v \end{cases} \qquad A = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 \cdots & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\dot{z} = Az + Bv$, (A, B) is controllable!

Is $z = T(x)$ a diffeomorphism?

<u>Lemma 4.1.1</u> (Isidori) $\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$ has a well-defined rel. deg. r, ~~then~~ the vectors

$$dh = [\frac{\partial h}{\partial x_1}, \cdots, \frac{\partial h}{\partial x_n}]$$

$$dL_f h = [\frac{\partial L_f h}{\partial x_1}, \cdots, \frac{\partial L_f h}{\partial x_n}]$$

$$\vdots$$

$$dL_f^{r-1} h = [\frac{\partial L_f^{r-1} h}{\partial x_1}, \cdots, \frac{\partial L_f^{r-1} h}{\partial x_n}]$$

are linearly independent!

$$\frac{\partial T}{\partial x} = \begin{bmatrix} dh(x) \\ dL_f h(x) \\ \vdots \\ dL^{n-1} h(x) \end{bmatrix} \Rightarrow \frac{\partial T}{\partial x} \text{ is nonsingular}$$

$r = n$

Example:

$$\begin{cases} \dot{x}_1 = x_2 + 2x_1^2 \\ \dot{x}_2 = x_3 + u \\ \dot{x}_3 = x_1 - x_3 \end{cases} \qquad n = 3$$

$y \neq x_1$, $\quad \dot{x}_1 = x_2 + 2x_1^2 \quad \ddot{x}_1 = \dot{x}_2 + \cdots = x_3 + u + \cdots$

$r = 2$

$\underline{y = x_3}$ $\quad \dot{y} = \dot{x}_3 = x_1 - x_3 \quad \ddot{y} = \dot{x}_1 - \dot{x}_3$

$$= x_2 + 2x_1^2 - (x_1 - x_2)$$

$$\dddot{y} = \dot{x}_2 + \alpha \dot{x}_1 = x_3 + u + \alpha x_1$$

$r = 3 = n$

$$\begin{aligned} z_1 &= x_3 \\ z_2 &= \dot{z}_1 \\ z_3 &= \ddot{z}_1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \underbrace{(4x_1 - 1)(x_2 + 2x_1^2)}_{\alpha(x)} + \underbrace{1}_{\beta(x)} u \end{aligned}$$

$$u = -(4x_1 - 1)(x_2 + 2x_1^2) + v$$

$$\Rightarrow \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = v \end{cases} \qquad v = -k_1 z_1 - k_2 z_2 - k_3 z_3$$

Result 2: $\dot{x} = f(x) + g(x)u$.

determine if it is feedback line: without searching for $h(x)$,

THM 13.2 (reading)

Drawbacks:

1. $\dot{x} = \theta x^3 + u$

$u = -\theta x^3 - kx,\ k>0$ $\Big\} \Rightarrow \dot{x} = -kx$

$u = -\hat{\theta} x^3 - kx \Rightarrow \dot{x} = \underline{(\theta - \hat{\theta})} x^3 - kx$

if $\hat{\theta} < \theta$, $\theta - \hat{\theta} > 0$

finite ~~escape~~ escape!

2. $\dot{x}_1 = x_2$

$\dot{x}_2 = -h(x_1) - u$, $h \in [0, \infty]$

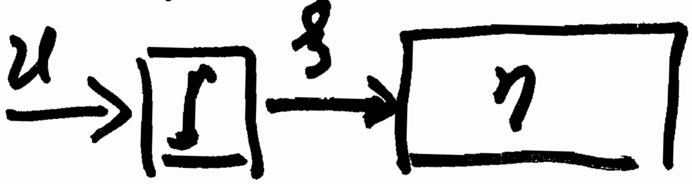
$u = h(x_1) - k_1 x_1 - k_2 x_2 \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k_1 x_1 - k_2 x_2 \end{cases}$

$u = -6(x_2)$, $6 \in [0, \infty]$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\underline{h(x_1)} - \underline{\sigma(x_2)} \end{cases} \Rightarrow \text{a.s.}$$

Backstepping (14.3)

Problem:
$$\dot{\eta} = f(\eta) + g(\eta)\xi$$
$$\dot{\xi} = u$$



Backstepping: Design a "virtual" control for η-subsystem, backstep it over the integrator to design "u".

Step 0: Design virtual input $\xi = \phi(\eta)$ s.t.

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) \text{ is g.a.s.}$$

with a Lyapunov function $V_0(\eta)$

$V_0(\eta)$ satisfies

$$\frac{\partial V_0}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \le -\underbrace{W(\eta)}_{\text{p.d.}}$$

Step 1: Define $z = \xi - \phi(\eta)$

$$V = V_0(\eta) + \frac{1}{2} z^2$$

$$\dot{V} = \frac{\partial V_0}{\partial \eta}\left(f(\eta) + g(\eta)\phi(\eta) + g(\eta)\xi - g(\eta)\phi(\eta)\right)$$

$$+ z(u - \dot{\phi})$$

$$\leq -W(\eta) + \frac{\partial V_0}{\partial \eta} g(\eta)\, z + z(u - \dot{\phi})$$

$$= -W(\eta) + z\left(\frac{\partial V_0}{\partial \eta} g(\eta) + u - \dot{\phi}\right)$$

$$u = -\frac{\partial V_0}{\partial \eta} g(\eta) + \dot{\phi} - kz, \quad k > 0$$

↓

$$\dot{V} \leq -W(\eta) - kz^2$$

$\because$ W is p.d., $\eta = 0, z = 0$ is g.a.s.

Example: $\begin{cases} \dot{\eta} = \eta^2 + \xi \\ \dot{\xi} = u \end{cases}$

Step 0: $\phi(\eta) = -\eta^2 - \eta$

when $\xi = \phi(\eta)$, $\dot{\eta} = -\eta$ $\quad V_0 = \frac{1}{2}\eta^2$ $\Downarrow$ $\dot{V}_0 = -\eta^2$

Step 1: $z = \xi - \phi(\eta)$

$$u = -\frac{\partial V_0}{\partial \eta} g(\eta) + \dot{\phi} - kz$$

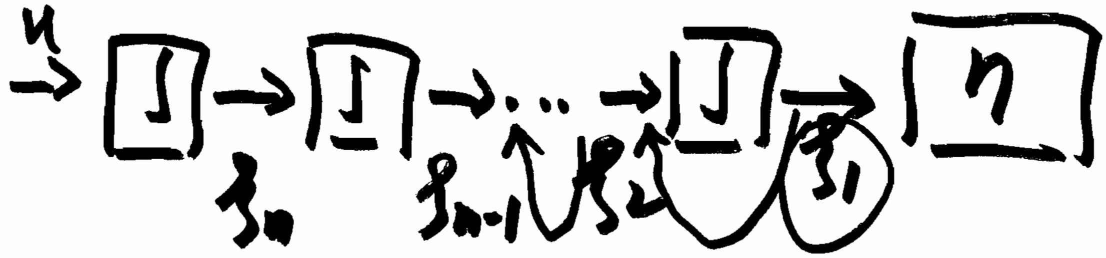
$V_0 = \frac{1}{2}\eta^2 \quad g(\eta) = 1 \quad \dot{\phi} = -2\eta\dot{\eta} - \dot{\eta}$

$$= (-2\eta - 1)\dot{\eta}$$

$$u = \underbrace{-k(\xi - \phi(\eta))}_{\Updownarrow\ -kz} \underbrace{-\eta}_{\Updownarrow\ -\frac{\partial V_0}{\partial \eta} g(\eta)} + \underbrace{(-1 - 2\eta)(\eta^2 + \xi)}_{\Updownarrow\ \dot{\phi}}$$

$\begin{cases} \eta = 0 \\ z = 0 \end{cases}$ is g. a. s.

More integrators:



Backstepping is not restricted to pure integrators:

$$\dot{\eta} = f(\eta) + g(\eta)\xi$$

$$\dot{\xi} = f_1(\eta, \xi) + g_1(\xi, \eta) u$$

If $g_1 \neq 0$, $u = \dfrac{1}{g_1(\xi, \eta)} \left(-f_1(\eta, \xi) + v \right)$

$$\dot{\xi} = v$$

Follow above procedures to design v.

More ~~syste~~ general system,

strictly feedback system:

$$\dot{\eta} = f_0(\eta) + g_0(\eta)\xi_1$$

$$\dot{\xi}_1 = f_1(\eta, \xi_1) + g_1(\eta, \xi_1)\xi_2$$

$$\dot{\xi}_2 = f_2(\eta, \xi_1, \xi_2) + g_2(\eta, \xi_1, \xi_2)\xi_3$$

$$\vdots$$

$$\dot{\xi}_r = f_r(\eta, \xi_1, \cdots, \xi_r) + g_r(\eta, \xi_1, \cdots, \xi_r)u$$