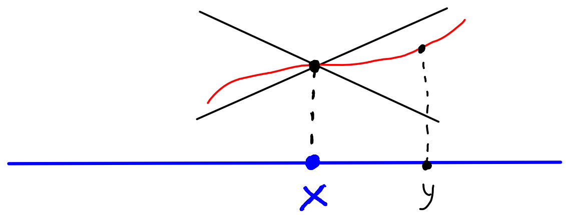
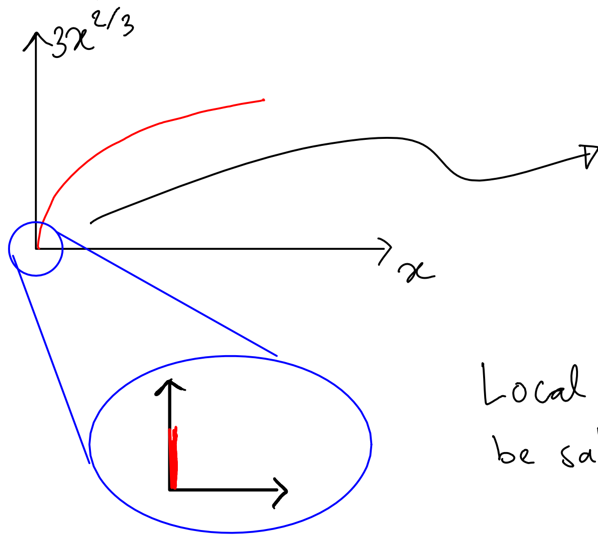
Existence & uniqueness of solutions

Local property --- (cont'd)

Lipschitz property: $\exists K$ such that $|f(y,t)-f(x,t)| \le K|y-x|$



Example: $\dot{x} = 3x^{2/3}$; $x(0) = 0$



$$\frac{dy}{dx} = 2x^{-1/3}$$

$$\lim_{x \downarrow 0} \frac{dy}{dx} = +\infty$$

Local Lipschitz conditions cannot be satisfied

Relation between Lipschitz property & continuity

Consider the system: $\dot{x} = f(x,t)$, $x \in \mathbb{R}^n$

Lemma 3.2: If $f(x,t)$ and $\frac{\partial f}{\partial x}(x,t)$ are continuous on $D \times [a,b]$, for some domain $D \subset \mathbb{R}^n$, then $f(x,t)$ is locally Lipschitz in x on $D \times [a,b]$

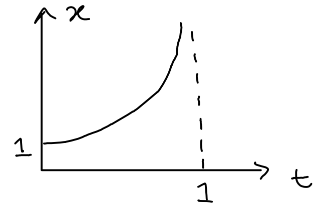
Global existence and uniqueness

Consider the system

$$\left.\begin{aligned} \dot{x} &= f(x,t), \quad x \in \mathbb{R}^n \\ x(0) &= x_0 \end{aligned}\right\} \quad \text{- - - - - } (*)$$

Example: $\dot{x} = x^2, \; x(0) = 1$

$$-\frac{1}{x} = t - 1 \quad \to \quad x(t) = \frac{1}{1-t}$$



Theorem for global existence and uniqueness

Assume that $f(x,t)$ is continuous in x and t, and that for every $T<\infty$ there exist K_T and h_T such that for all $t \in [0,T]$

$$|f(x,t) - f(y,t)| \le K_T |x-y|, \; \forall x,y \in \mathbb{R}^n$$

$$|f(x_0,t)| \le h_T$$

then $(*)$ has exactly one solution on $[0,T]$ for all $T < \infty$

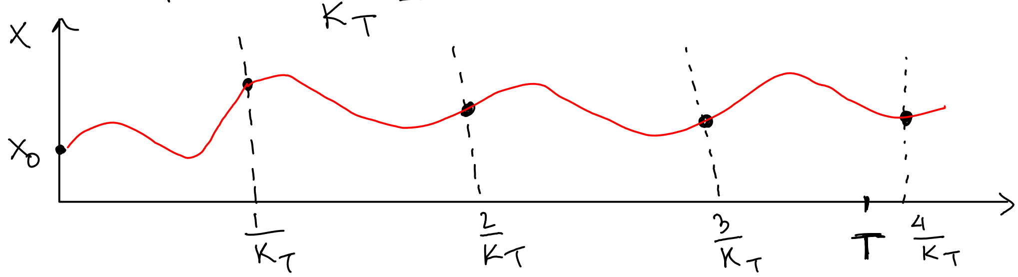
Proof: Consider local existence and uniqueness thm; take the limit of $r \to \infty$. We know that there is a unique solution for $t \in [0,\delta]$ where

$$\delta < \min\left[\frac{1}{K_T}, \frac{r}{K_T r + h_T}\right]$$

Since $\displaystyle\lim_{r\to\infty} \frac{r}{K_T r + h} = \frac{1}{K_T}$, we have a unique solution

for $t \in [0,\delta]$, $\delta < \frac{1}{K_T}$. If $\frac{1}{K_T} \ge T$, then we are done.

Otherwise, apply the local thm repeatedly to get m intervals. Stop when $\frac{m}{K_T} \ge T$.



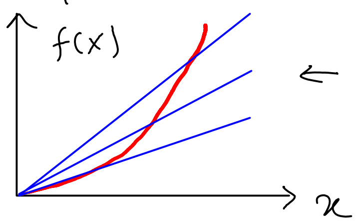
Note: The constant h_T needs to be changed for every interval. However, observe that we only need the existence of h_T, not its value. The existence of h_T for the second interval is guaranteed by:

$$|f(x(\tfrac{1}{K_T}), t)| = |f(x(\tfrac{1}{K_T}, t) - f(x_0, t) + f(x_0, t)|$$

$$\leq |f(x(\tfrac{1}{K_T}, t) - f(x_0, t)| + |f(x_0, t)|$$

$$\leq K_T \cdot \underbrace{\|x(\tfrac{1}{K_T}) - x_0\|}_{\text{finite}} + h_T$$

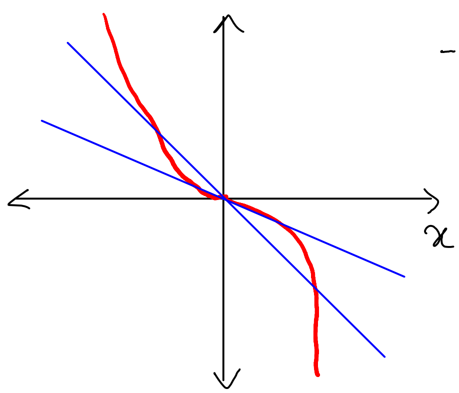
Example: $f(x) = x^2$ is not globally Lipschitz



$\leftarrow$ Pictorial; Formal $\rightarrow$ $x^2 - Kx > 0$ if $x > K$

Note: Lipschitz property is sufficient, not necessary.

Example: $\dot{x} = -x^3$; $x(0) = x_0$



$-x^3 + kx = x(k - x^2) < 0$ if $x > \sqrt{k}$

thus non-Globally Lipschitz

However, there is a global solution:

$$\frac{1}{2} x^{-2} = t + \frac{1}{2} x_0^{-2}$$

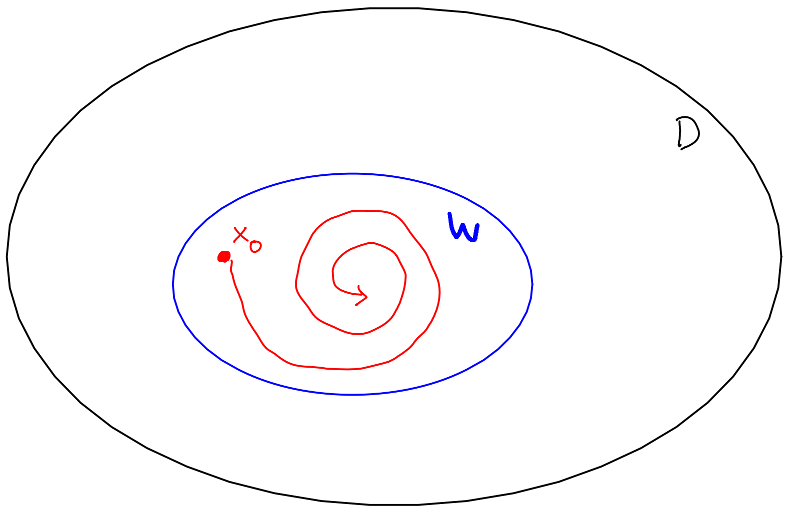
$$x(t) = \sqrt{\frac{1}{2t + x_0^{-2}}}$$

Theorem 3.3: Let $f(x,t)$ be continuous in t and locally Lipschitz in x for all $t \geq t_0$, and all x in a domain $D \subset \mathbb{R}^n$. Let W be a compact (closed and bounded) subset of D with $x_0 \in W$, and suppose that it is known that the solution of

$$\dot{x} = f(x,t) \; ; \; x(t_0) = x_0$$

remains in W, Then, there is a unique solution that is defined for all $t \geq t_0$.

Sketch:



Intuitively: As long as $x(t)$ does not "blow out", local Lipschitz property is enough to ensure unique solution for $t \to \infty$.