Lyapunov Stability

Consider the system : $\dot{x} = f(x,t)$, $x \in \mathbb{R}^n$

A ~~state~~ state $\bar{x}_0 \in \mathbb{R}^n$ is an equibrium if :

$$\boxed{f(\bar{x}_0, t) = 0 \quad , \forall t}$$

Stability of equilibria : without any loss of generality , assume $\bar{x}_0 = 0$

Classification of stability :

- The equilibrium is **stable** if $\forall \varepsilon > 0, \exists \delta > 0$ such that :

$$\| x(0) \| < \delta \rightarrow \| x(t) < \varepsilon \| \quad \text{for all } t > 0$$

sketch!

- The equilibrium is **unstable** if it is not stable

- The equilibrium is **asymptotically stable** if $\exists \delta > 0$ such that

$$\| x(0) \| < \delta \Rightarrow \lim_{t \to \infty} \| x(t) \| = 0$$

- The equilibrium is **exponentially stable** if $\exists \delta > 0$ such that

$$\| x(0) \| < \delta \Rightarrow \exists a, b > 0 \text{ s.t } \| x(t) \| \leq a \cdot e^{-bt}$$

Note : Exponential stability $\Rightarrow$ asymptotic stability $\Rightarrow$ stability

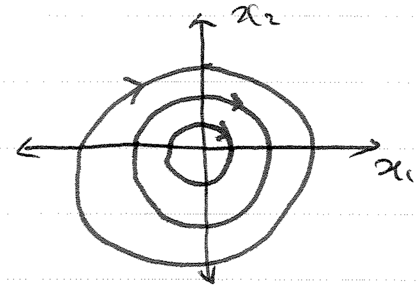
Example : $\dot{x} = -x^3$, $x \in \mathbb{R}$

$x(0) = x_0$.

Solving the diff. eq , we get : $x(t) = \text{sign}(x_0) \sqrt{\dfrac{x_0^2}{1 + 2x_0^2 t}} \sim \dfrac{1}{\sqrt{t}}$

The origin is asymptoticall stable but not exponentially stable.

Example: $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



Stable but not asymp. stable

Lyapunov Theory: Use potential like functions to infer about stability

Consider: $\dot{x} = f(x)$, $f(0) = 0$, $x \in \mathbb{R}^n$

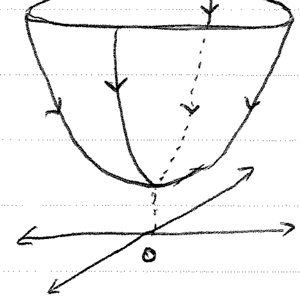
Thm: Suppose that $\exists$ a function $V(x)$ such that $\forall x \in D$ $(D \ni 0)$

(1) $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$

(2) $\frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) \leq 0$, $\forall x \in D$, or

(2a) $\frac{dV}{dt} < 0$, $\forall x \neq 0$, $x \in D$

(3) $V(\cdot)$ is continuously differentiable



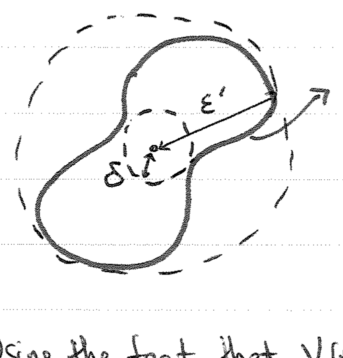
Then:

A) (1)+(2)+(3) $\rightarrow$ the origin is stable

B) (1)+(2a)+(3) $\rightarrow$ the origin is asymp. stable

Proof: A) $\forall \varepsilon > 0$, $\exists \delta$ s.t. $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon$, $\forall t > 0$

Pick $\varepsilon' \leq \varepsilon$ such that $B(0, \varepsilon') \subset D$ (guarantees that the properties are valid)



$\{x \mid V(x) \leq \beta\}$

Pick $\beta > 0$ such that $\{x \mid V(x) \leq \beta\}$ is contained in $B(0, \varepsilon')$, then pick $\delta < \min\limits_{\{x \mid V(x) = \beta\}} \|x\|$

Using the fact that $V(t)$ is nonincreasing, and:

$\left.\begin{array}{l} \|x\| < \delta \Rightarrow V(x) < \beta \\ \|x\| \geq \varepsilon' \Rightarrow V(x) \geq \beta \end{array}\right\}$ we infer that A) is true.

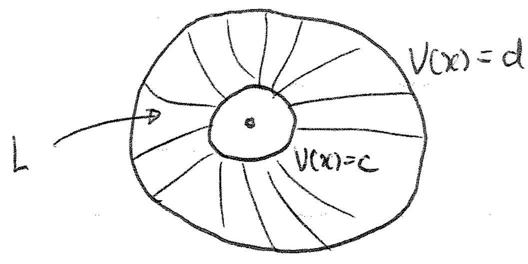
B) $\exists \delta$ s.t. $\lim\limits_{t \to \infty} \|x(t)\| = 0$ if $\|x(0)\| < \delta$. Pick δ such that $B(0, \delta) \subset D$.

It suffices to show that $\lim\limits_{t \to \infty} V(t) = 0$

Since $V(t)$ is bounded from below, and nonincreasing, we know that

$$\lim_{t \to \infty} V(t) = c \qquad \text{(why?)} \qquad \text{for some } c > 0.$$

Now, we need to show that $c = 0$. Suppose that $c \neq 0$, take any $d > c$ and define $L = \{x \mid c \leq V(x) \leq d\}$, by definition $\exists T > 0$ such that $\forall t > T$, $x(t) \in L$.



Consider $\min_{x \in L} \dot{V}(x) \triangleq a$. We know that $a < 0$. We then have that

$\forall t > T$:

$$V(x(t)) = V(x(T)) + \int_T^t \dot{V}(x(\tau))d\tau \leq d + a(t-T)$$

However, that means for $t > T + \frac{c-d}{a}$, $V(x(t)) < c$ which is a contradiction.

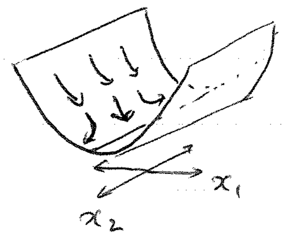
Definition: A function $V(x)$ is <u>positive definite</u> if: $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. $V(x)$ is <u>positive semi-definite</u> if $V(0) = 0$ and $V(x) \geq 0$.
Similarly: <u>negative definite</u> and <u>negative semi-definite</u>.

Note: The previous theorem requires that $V(x)$ is positive definite, not positive semidefinite.

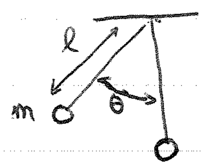
Example: $\dot{x}_1 = -x_1^3$
$\dot{x}_2 = x_2^2$

Define a positive semidefinite function $V(x) = x_1^2$

$\frac{dv}{dt} = 2x_1 \cdot -x_1^3 = -2x_1^4 \leq 0$ However the origin is <u>not stable</u>



Example: Pendulum system



$$m\ell^2 \cdot \frac{d^2\theta}{dt^2} = -mg\ell \sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell} \sin\theta$$

Define: $\left.\begin{array}{l}\theta = x_1 \\ \dot\theta = x_2\end{array}\right\}$ $\begin{array}{l}\dot x_1 = x_2 \\ \dot x_2 = -\frac{g}{\ell}\sin x_1\end{array}$

Show that $\begin{pmatrix}\theta = 0 \\ \dot\theta = 0\end{pmatrix}$ is stable: Choose total energy as Lyapunov function:

$V(x) = \cancel{mg\ell \sin\theta}\ mg\ell(1-\cos x_1) + \frac{1}{2} m\ell^2 x_2^2$

Notice that: $V(0) = 0$, and $V(x) > 0$ if $x \neq 0$ in some domain around $x = 0$

$\frac{dv}{dt} = -mg\ell \sin x_1 \cdot x_2 + m\ell^2 x_2 \cdot (-\frac{g}{\ell}\sin x_1)$

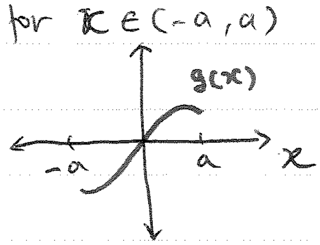
$= 0$

Thus $\theta = 0, \dot\theta = 0$ is (locally) stable.

Example: $\dot x = -g(x)$, $x \in \mathbb{R}$ such that:

- $g(x)$ is Locally Lipschitz in $(-a, a)$
- $g(0) = 0 \rightarrow x = 0$ is an equilibrium
- $x \cdot g(x) > 0$, for $x \in (-a, a)$

Sketch:



Show that $x = 0$ is asymp. stable.

Idea: Integrate $g(x)$ to get a potential function:

$$V(x) = \int_0^x g(x)\,dx \,, \quad x \in (-a, a)$$ Obviously: $V(0) = 0$
$V(x) \neq 0$ if $x \neq 0$

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} = g(x) \cdot -g(x) = -g(x)^2 < 0 \quad \text{if } x \neq 0$$

Thus the origin is asymptotically stable

Caveat: Does not always work in higher dimensional spaces.