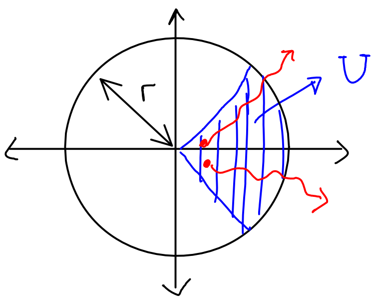
Instability result

Consider the system: $\dot{x} = f(x)$, $x \in \mathbb{R}^n$

$f(0) = 0$, $f(\cdot)$ is Lipschitz

Suppose that $V(x)$ is continuously differentiable in $D \subset \mathbb{R}^n$, and $V(0) = 0$. Further, suppose that for any $\delta > 0$, there is a point $x_0 \in B(0,\delta)$ such that $V(x_0) > 0$. Take an $r > 0$ such that $B(0,r) \subset D$.

Define: $U = \{x \in B(0,r) \mid V(x) > 0\}$



Thm: If $\frac{dV}{dt} > 0$ in U then the equilibrium is not stable.

Example:
$$\dot{x}_1 = x_1 + g_1(x)$$
$$\dot{x}_2 = -x_2 + g_2(x)$$
g_1 and g_2 are Lipschitz and satisfy

$$|g_1(x)| \le k\|x\|^2 \text{ and } |g_2(x)| \le k\|x\|^2$$

Take $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$. See that on $(x_2 = 0)$, $V(x)$ is positive while arbitrarily close to the origin.

$$\frac{dV}{dt} = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

But $|x_1 g_1(x) - x_2 g_2(x)| \le 2k\|x\|^3$, thus:

$\frac{dV}{dt} \ge \|x\|^2 - 2k\|x\|^3$, for $\|x\|$ small enough, $\frac{dV}{dt} > 0$. The origin is unstable.

Global Asymptotic Stability

The origin is globally asymptotically stable if $\forall\, x(0) \in \mathbb{R}^n$,
$$\lim_{t \to \infty} x(t) = 0$$

Note: GAS $\Rightarrow$ unique equilibrium

Thm: Suppose that $V(x)$ is such that:

(1) It is continuously differentiable

(2) $V(0)=0$ and $V(x)>0$ if $x\neq 0$

(3) $\frac{dV}{dt}<0$ if $x\neq 0$

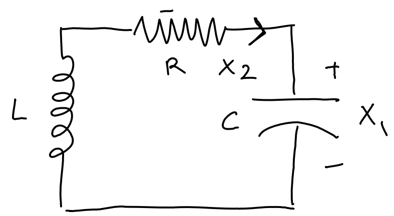
everywhere in $\mathbb{R}^n$, and

(4) $\|V(x)\|\to\infty$ if $\|x\|\to\infty$

Why (4)? Needs every level set to be bounded (Recall the proof of the local theorem)

Invariance Principle

Consider the following system:



$L=R=C=1$

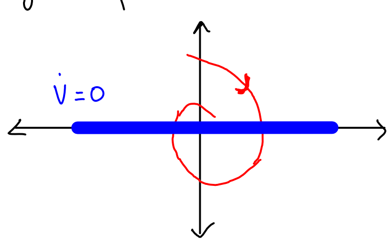
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2$$

Potential Energy $= V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ (+ def)

$\dot{V} = x_1x_2 + x_2(-x_1 - x_2)$

$= -x_2^2$ (− semi def)

Therefore, the equilibrium is stable (not necessarily asymp. stable). However we can easily compute that it is actually asymp. stable (look at the eigenvalues)



Theorem: Suppose that $\Omega \subset D$ is an invariant set, and $V(x)$ is continuously differentiable in D and $\dot{V}(x) \leq 0$ in Ω. Define $E \subset \Omega$ as

$$E = \{x \in \Omega \mid \dot{V}(x) = 0\},$$

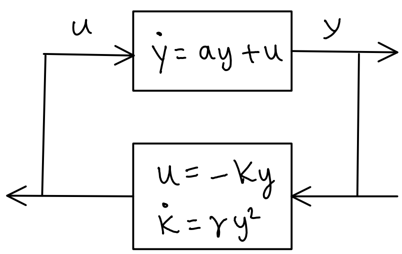
and M as the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Apply to previous example: $\Omega = B(0, r)$ for some $r > 0$

$E = \{(x_1, x_2) \mid x_2 = 0\}$

$M = (0,0)$

Example: Adaptive control scheme:



$\gamma > 0$

Consider: $x_1 = y$

$x_2 = k$

$\dot{x}_1 = a x_1 - x_1 x_2$

$\dot{x}_2 = \gamma x_1^2$

Equilibria at $x_1 = 0$. Take Lyapunov function

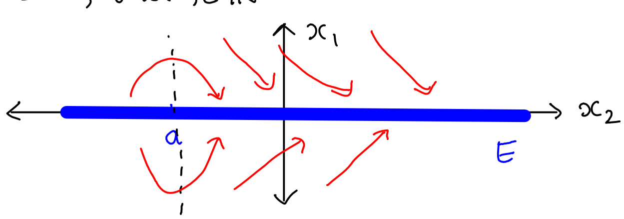
$V(x) = \frac{1}{2} x_1^2 + \frac{1}{2\gamma} (x_2 - b)^2 \leftarrow$ radially unbounded

$$\frac{dV}{dt} = x_1 (a x_1 - x_1 x_2) + \frac{1}{\gamma} (x_2 - b) \gamma x_1^2$$

$$= x_1^2 (a - x_2) + x_1^2 x_2 - b x_1^2 = (a - b) x_1^2$$

Choose b large enough s.t $b > a$, then $\dot{V}$ is $-$semidef. The set $E = \{x \mid \dot{V}(x) = 0\} = \{(x_1, x_2) \mid x_1 = 0\}$, which is invariant. Therefore

$$\lim_{t \to \infty} x(t) \in E, \quad \forall x(0) \in \mathbb{R}^2$$



Linear Systems and Quadratic Lyapunov Functions

Consider a linear system: $\dot{x} = Ax \quad , x \in \mathbb{R}^n$

Propose a Lyapunov function $V(x) = x^T P x$, with P a + def matrix.

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x \triangleq x^T Q x$$

with: $Q = A^T P + PA$. If Q is negative definite, then the system is stable.

Theorem: If A is stable, then for any $Q < 0$, the Lyapunov equation

$$A^T P + PA = Q$$

has a unique pos. definite matrix P as solution.

Linearization (Lyapunov Indirect Method)

Consider the system: $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, $f(0) = 0$.

f is continuously differentiable in D

Theorem: Define $A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=0}$. Then:

a) The origin is asymp. stable if $\mathrm{Re}\,\lambda_i < 0$ for all eigenvalues of A

b) The origin is unstable if $\mathrm{Re}\,\lambda_i > 0$ for at least one eigenvalue of A

Proof: a) Take a $P > 0$ such that $A^T P + PA = Q < 0$

Define: $V(x) = x^T P x$, $\dot{V} = \dot{x}^T P x + x^T P \dot{x}$

$$= [x^T A^T + g^T(x)] P x + x^T P (Ax + g(x))$$

$g(x)$: higher order terms

$$= \underbrace{x^T Q x}_{2^{nd}\text{ order}} + \underbrace{g^T(x) P x + x^T P g(x)}_{\text{higher order}}$$

< 0 if $x \neq 0$ and $\|x\|$ small enough

Thus 0 is asymp. stable

Example: Damped Pendulum: $\dot{x}_1 = x_2$

$\dot{x}_2 = -a \sin x_1 - b x_2$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -a\cos x_1 & -b \end{bmatrix} \rightarrow \text{evaluate at the origin: } \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

$\lambda_{1,2} = -\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 - 4a}$, thus the origin is asymp. stable