ECSE 6420: Nonlinear Control Systems

Homework set 1. Due date: 5 Feb 2009

Points: Problem 1 = 10pts, Problem 2 = 5+10+10+5 pts, Problem 3 = 20pts, Problem 4 = 5+10 pts, Problem 5 = 10+15 pts

1. Consider an autonomous dynamical system

$$\dot{x} = f(x),\ x(0) = x_0,\ x \in \mathbb{R}^n. \tag{1}$$

Assume that $f(\cdot)$ is such that (1) has a unique solution for all positive time[1]. Such a system can be thought of as a map from x_0 to the solution of (1). That is, if we denote the map as $\mathcal{H}$, then $\mathcal{H}(x_0)(t)$ is the differentiable solution of (1). The system is **linear** if and only if $\mathcal{H}$ is a linear map, i.e.

$$\forall x_1, x_2 \in \mathbb{R}^n, \forall a_1, a_2 \in \mathbb{R}, \mathcal{H}(a_1 x_1 + a_2 x_2) = a_1 \mathcal{H}(x_1) + a_2 \mathcal{H}(x_2). \tag{2}$$

Prove that $f(x)$ can always be written as Ax, for some constant matrix A.

2. The predator-prey model that we discussed in class assumes unlimited resources that enable the prey to grow unboundedly in the absence of the predators. A more realistic model that model competition within the same species is posed below:

$$\dot{x} = ax - bxy - \lambda x^2, \tag{3}$$

$$\dot{y} = cxy - dy - \mu y^2. \tag{4}$$

(a) Prove that the first quadrant $\{(x, y) | x \geq 0, y \geq 0\}$ is invariant.
(b) Specify all the equilibria in the first quadrant and simulate for $a = b = c = d = 1$, $\lambda = 0.5$, $\mu = 1$. Use Hartman-Grobman Theorem (when applicable) to classify the types of the equilibria
(c) Specify all the equilibria in the first quadrant and simulate for $a = b = c = d = 1$, $\lambda = 2$, $\mu = 1$. Use Hartman-Grobman Theorem (when applicable) to classify the types of the equilibria
(d) Discuss the difference in the behavior of the system in (b) and (c).

3. Hartman-Grobman Theorem asserts that local linearization of a nonlinear system $\dot{x} = f(x)$ around an equilibrium x_0 can qualitatively describe the local nonlinear dynamics, provided that the eigenvalues of the Jacobian matrix at x_0 exclude the imaginary axis. This is because when there

[1] Later in the course we are going to study conditions on $f(\cdot)$ such that this is true.

are some eigenvalues on the imaginary axis, higher order dynamics can become the dominant factor. Consider the following two systems:

$$\Sigma_1 : \dot{x} = x^2 \text{ and } \Sigma_2 : \dot{x} = -x^3.$$

Both systems have an equilibrium at the origin, and their linearizations around this equilibrium are identical. The eigenvalue of the Jacobian matrix is 0. Observe that the origin is an unstable equilibrium for Σ_1, but stable for Σ_2.
Task: Construct an example with two second-order nonlinear systems, Σ_1 and Σ_2, such that their linearizations around an equilibrium are identical. However, design them such that the equilibrium is a **stable focus** for Σ_1 and **unstable focus** for Σ_2.

4. (Problem 2.22) Consider the system

$$\begin{aligned} \dot{x}_1 &= ax_1 - x_1 x_2, \\ \dot{x}_2 &= bx_1^2 - cx_2, \end{aligned}$$

where a, b, c are positive constants with $c > a$. Let $D = \{x \in \mathbb{R}^2 \mid x_2 \geq 0\}$.
(a) Show that D is invariant
(b) Show that there can be no periodic orbit that passes through any point in D. Notice that this is a stronger statement than "D cannot contain any periodic orbit".

5. (Normed linear spaces) Show that the following linear space and norm form a normed linear space:
(a) The space of real sequences $\{x_k\}_{k=1,\ldots,\infty}$ with the norm

$$\|x\| := \sum_{k=1}^{\infty} |x_k| .$$

(b) The space of continuous real functions $x : [0, T] \to \mathbb{R}$, with the norm

$$\|x\| := \sqrt{\int_0^T x^2(t)\, dt}.$$