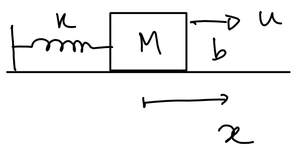
Minimal Realization: (A, B) controllable

(A, C) observable $\Leftrightarrow$ (A^T, C^T) controllable

Example: Mass-spring-damper



$x_1 = x$

$x_2 = \dot{x}$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}(u - kx_1 - bx_2) \\ y &= x_1 \end{aligned}\right\} \quad A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{pmatrix} ; \; B = \begin{pmatrix} 0 \\ \frac{1}{M} \end{pmatrix}$$

$$C = [1 \quad 0]$$

$$\left.\begin{aligned} \dim \mathcal{C} &= \text{rk} \begin{bmatrix} 0 & \frac{1}{M} \\ \frac{1}{M} & -\frac{b}{M^2} \end{bmatrix} = 2 \\ \dim \mathcal{O} &: \text{rk} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2 \end{aligned}\right\} \text{ minimal realization}$$

$$Y(s) = C(sI - A)^{-1} B \, U(s) = [1 \quad 0] \begin{bmatrix} s & -1 \\ +\frac{k}{M} & s + \frac{b}{M} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$$

$$= \frac{1/m}{\left(s^2 + s\frac{b}{m} + \frac{k}{M}\right)}$$

Poles at the roots of $\mathcal{X}(s) = \det(sI - A)$

We can speed up the response (or slow down) by using pole-placement

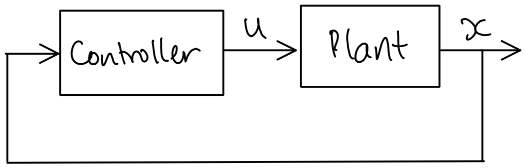
Define: $u = -Kx + v$

$\dot{x} = (A - BK)x + Bv$

$y = Cx$

Notice that new poles are the roots of $\det(sI - A + BK)$

Implication of observability:



State Feedback Control

In many cases, it is not practical to assume that all states can be measured. Often, only y and u are available for control.

Observer design: use y and u to estimate x

Plant: $\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\}$ Assume that (A, C) is observable

Observer: $\dot{\hat{x}} = A\hat{x} + Bu + F(y - \hat{y})$

$\hat{y} = C\hat{x} + Du$

Define: $e \triangleq x - \hat{x} \rightarrow \dot{e} = Ae - FCe$

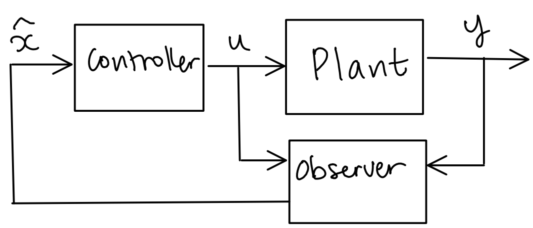
$= (A - FC)e$

The eigenvalues of $(A - FC)$ are the same as those of $(A^T - C^T F^T)$

Recall that (A, C) observable means (A^T, C^T) is controllable, which means that F can be designed to make $\hat{x}$ converge to x arbitrarily fast.

(recall the pole placement theorem)

Close Loop Control using Observer



Output Feedback

We need to design both the controller simultaneously

Plant: $\dot{x} = Ax + Bu$, $y = Cx + Du$ } assume minimal realization

Observer: $\dot{\hat{x}} = A\hat{x} + Bu + F(y - \hat{y}) = A\hat{x} + Bu + FCx - FC\hat{x}$

$\hat{y} = C\hat{x} + Du$

Controller: $u = -K\hat{x} + v$

Close-loop dynamics:

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ FC & A-BK-FC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} B \\ B \end{pmatrix} v$$

State transformation: $\begin{pmatrix} e \\ \hat{x} \end{pmatrix} = \underbrace{\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}}_{T} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$

$$T^{-1} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} \dot{e} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} A & -BK \\ FC & A-BK-FC \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} e \\ \hat{x} \end{pmatrix}$$

$$= \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} A & A-BK \\ FC & A-BK \end{pmatrix} \begin{pmatrix} e \\ \hat{x} \end{pmatrix} = \begin{pmatrix} A-FC & 0 \\ FC & A-BK \end{pmatrix} \begin{pmatrix} e \\ \hat{x} \end{pmatrix}$$

Notice the block triangular structure. The closed loop poles are the poles of $(A-FC)$ and the poles of $(A-BK)$

This is called the separation principle: the controller K and the observer gain F can be designed separately.

Multivariable aspects

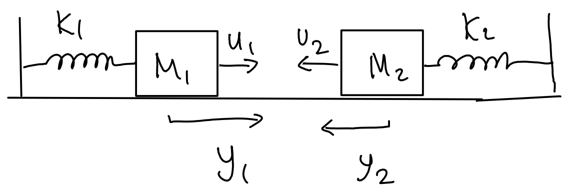
$$\dot{x} = Ax + Bu \qquad , x \in \mathbb{R}^n , u \in \mathbb{R}^m , y \in \mathbb{R}^\ell$$
$$y = Cx + Du$$

- If (A, B) is controllable, but none of (A, B_i) is controllable, then some of the poles cannot be moved by using only one input. (Which?)

- If (A, C) is controllable, but none of (A, C_i) is observable, then some states cannot be observed arbitrarily fast by using one output (which?)

- Another implication of partial observability $\rightarrow$ some outputs might not "see" all the poles.

(Trivial) Example:



y_1 does not "see" the poles associated with the oscillation of M_2, and vice versa. u_1 cannot be used to move the poles associated with M_2, and vice versa

Pole Direction:

$$Y(s) = C(sI - A)^{-1} B \; U(s)$$

Decompose A using eigenvectors: $A v_i = v_i \lambda_i$

$$AV = V\lambda \rightarrow A = V \lambda V^{-1}$$

V is the matrix of right eigenvectors of A. But observe that

$$V^{-1} A = \lambda V^{-1}$$

The rows of V^{-1} are the left-eigenvectors of A.

$$(sI - A) = V \begin{pmatrix} s-\lambda_1 & \cdots & & \vdots \\ \vdots & s-\lambda_2 & & \vdots \\ \vdots & \cdots & \ddots & s-\lambda_n \end{pmatrix} V^{-1}$$

$$(sI - A)^{-1} = V \begin{pmatrix} \frac{1}{s-\lambda_1} & \cdots & & \\ \vdots & \frac{1}{s-\lambda_2} & & \\ \vdots & & \ddots & \frac{1}{s-\lambda_n} \end{pmatrix} V^{-1}$$

$$Y(s) = \underbrace{CV}_{l \times n} \begin{pmatrix} \frac{1}{s-\lambda_1} & \cdots & & \\ \vdots & \frac{1}{s-\lambda_2} & & \\ \vdots & & \ddots & \frac{1}{s-\lambda_n} \end{pmatrix} \underbrace{V^{-1}B}_{n \times m} \; U(s)$$

* The i-th row of $V^{-1}B$ represents the influence of each input to the pole at λ_i.
* The i-th column of CV represents the influence of the pole at λ_i to each output

The i-th row of $V^{-1}B$ is $w_i B$, where w_i is the i-th left eigenvector
The i-th column of CV is Cv_i, where v_i is the i-th right eigenvector