Geometric Theory of Multivariable Control

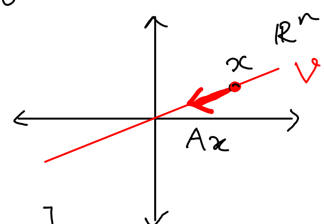
Consider the state-space representation:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$
$$y = Cx, \quad y \in \mathbb{R}^\ell$$

A subspace $\mathcal{V} \subset \mathbb{R}^n$ is an <u>invariant subspace</u> (or A-invariant subspace) if: $\forall x \in \mathcal{V}$, $Ax \in \mathcal{V}$

Shorthand notation: $A\mathcal{V} \subset \mathcal{V}$

Interpretation: Set the input $u=0$. Any initial state in $\mathcal{V}$ results in a trajectory that remains in $\mathcal{V}$.



Example: Suppose that $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $n=2$.

The space $\mathcal{V} = \operatorname{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is invariant. This is because for any vector $x \in \mathcal{V}$,

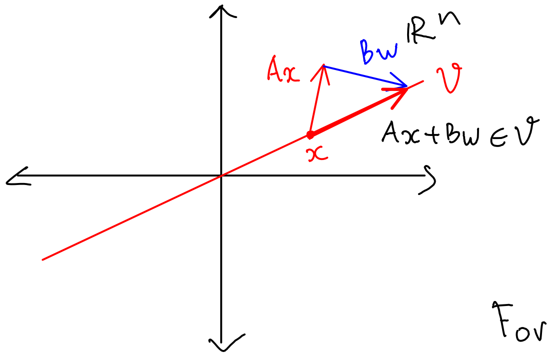
$$x = \begin{bmatrix} k \\ 0 \end{bmatrix}, \quad Ax = \begin{bmatrix} -k \\ 0 \end{bmatrix} \in \mathcal{V}.$$

Note: You can check invariance by checking the product of A and a set of basis vectors of $\mathcal{V}$ (from linearity)

Note: Any eigenvector of A spans a 1-dimensional invariant space of A.

A subspace $\mathcal{V} \subset \mathbb{R}^n$ is a <u>controlled-invariant subspace</u> (or (A,B)-invariant subspace) if: $\forall x \in \mathcal{V}$, there exists a control vector $w \in \mathbb{R}^m$ such that $Ax + Bw \in \mathcal{V}$ (w generally depends on x)

$$\boxed{\mathcal{V} + \mathcal{W} \triangleq \{ z \mid \exists x \in \mathcal{V}, y \in \mathcal{W}, \; z = x + y \}}$$



Shorthand notation:

$$A\mathcal{V} \subset \mathcal{V} + \text{im } B$$

Read: Any element of $A\mathcal{V}$ can be written as a sum of an element of $\mathcal{V}$ and an element of $\text{im } B$. Thus:

For any $x \in \mathcal{V}$, there exist $q \in \mathcal{V}$ and $\hat{w} \in \mathbb{R}^m$ such that $Ax = q + B\hat{w}$

$$Ax - B\hat{w} = q \text{ , or}$$

$$Ax + Bw = q \text{ , by renaming } w = -\hat{w}.$$

Interpretation: we can always find an input that makes the space $\mathcal{V}$ invariant

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $(n=2, m=1)$

The space $\mathcal{V} = \text{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is controlled invariant. To see this, take any

$x \in \mathcal{V}$, $x = \begin{bmatrix} k \\ k \end{bmatrix} \rightarrow Ax = \begin{bmatrix} -k \\ k \end{bmatrix}$, take $w = -2k$, then

$$Ax + Bw = \begin{bmatrix} -k \\ k \end{bmatrix} + \begin{bmatrix} 0 \\ -2k \end{bmatrix} = \begin{bmatrix} -k \\ -k \end{bmatrix} \in \mathcal{V}$$

Note: Again, you can check by taking a set of basis vectors in $\mathcal{V}$. In fact, this will reveal an interesting relation:

Suppose that $\mathcal{V}$ is a p-dimensional subspace and $(x_1, x_2, \dots, x_p)$ is a set of basis vectors of $\mathcal{V}$.

Suppose that for $i = 1 \dots p$, there exist $w_i \in \mathbb{R}^m$ such that

$$Ax_i + Bw_i = q_i \in \mathcal{V} \text{ , } i = 1 \dots p.$$

Any other vector $x \in \mathcal{V}$ can be written as:

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_p x_p \text{ , } \alpha_{1\dots p} \in \mathbb{R}$$

Define $w \triangleq \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_p w_p \in \mathbb{R}^m$, then

$$Ax + Bw = A \cdot \sum_{i=1}^{p} \alpha_i x_i + B \sum_{i=1}^{p} \alpha_i w_i$$

$$= \sum_{i=1}^{p} \alpha_i q_i \in \mathcal{V} //$$

Notice that w can be computed as a linear feedback:

$$w = [w_1 \vdots w_2 \vdots \cdots \vdots w_p][\alpha_1 \ \alpha_2 \ \cdots \alpha_p]^T$$

$$x = [x_1 \vdots x_2 \vdots \cdots \vdots x_p][\alpha_1 \ \alpha_2 \cdots \alpha_p]^T, \text{ thus}$$

$$w = \underbrace{[w_1 \vdots w_2 \vdots \cdots \vdots w_p][x_1 \vdots x_2 \vdots \cdots \vdots x_p]^\dagger}_{F} x$$

$$\boxed{w = Fx}$$

$[x_1 \vdots x_2 \vdots \cdots \vdots x_p]^\dagger$ is the left pseudo inverse of $[x_1 \vdots x_2 \vdots \cdots \vdots x_p]$

$$X^\dagger = (X^T X)^{-1} X^T$$

This also means that $\mathcal{V}$ is an invariant space of the closed-loop system $\dot{x} = (A + BF)x$

In the previous example: $w = -2k \cdot \left([k \ \ k] \begin{bmatrix} k \\ k \end{bmatrix} \right)^{-1} [k \ \ k] \ x$

$$= \frac{-2k}{2k^2} [k \ \ k] \ x = -[1 \ \ 1] \ x$$

Lemma: If $\mathcal{V}$ and $\mathcal{W}$ are both (A,B) invariant subspaces, then $\mathcal{V} + \mathcal{W}$ is also (A,B) invariant

Proof: For any $x \in \mathcal{V} + \mathcal{W}$, there exist $x_1 \in \mathcal{V}$, and $x_2 \in \mathcal{W}$ such that $x = x_1 + x_2$, However since $\mathcal{V}$ and $\mathcal{W}$ are both (A,B) invariant, we have $w_1, w_2 \in \mathbb{R}^m$ such that

$$Ax_1 + Bw_1 \in \mathcal{V}; \quad Ax_2 + Bw_2 \in \mathcal{W}$$

Define $w = w_1 + w_2$, then

$$Ax + Bw = Ax_1 + Ax_2 + Bw_1 + Bw_2 \in \mathcal{V} + \mathcal{W}$$

Disturbance Decoupling Problem (DDP)

Consider the state space representation with disturbance:

$$\dot{x} = Ax + Bu + Gd \quad , d \in \mathbb{R}^d$$
$$y = Cx$$

Problem: we want to find a linear feedback control $u = -Kx$ such that the disturbance d is perfectly decoupled from the system, i.e the TF from d to y is 0.

Recall that the space $\{x \in \mathbb{R}^n \mid Cx = 0\}$ is called $\ker C$, which is a space of states corresponding to zero output (not to be confused with unobservable space)

Idea: Design the linear feedback such that there is a subspace $\mathcal{V}$:

$$\mathcal{V} \subset \ker C \quad \text{.....} (*)$$
$$(A - BK)\mathcal{V} \subset \mathcal{V} \quad \text{.....} (**)$$
$$\operatorname{im} G \subset \mathcal{V} \quad \text{.....} (***)$$

Theorem: There exists a unique maximal controlled invariant subspace in $\ker C$. We denote it by $\mathcal{V}^*$

Proof: There is always a CI space with dimension 0, i.e the origin.

Perform the following procedure: $V_0 \triangleq \{0\}$

Set $k = 0$;

If there is a $\mathcal{W} \subset \ker C$, $\mathcal{W}$ is CI, $\mathcal{W} \not\subset V_k$, then

$\{ V_{k+1} = V_k + \mathcal{W};$

$k = k+1;$

$\}$

The procedure above will terminate after at most $\dim(\ker C)$ steps.

the final value is $\mathcal{V}^*$

Theorem: DDP is solvable if and only if $\operatorname{im} G \subset \mathcal{V}^*$

How to calculate $\mathcal{V}^*$?

Define: $A^{-1}\mathcal{V} \triangleq \{x \mid Ax \in \mathcal{V}\}$

Iteration: $\mathcal{V}_0 = \ker C$

$$\mathcal{V}_{i+1} = \mathcal{V}_i \cap A^{-1}(\mathcal{V}_i + \operatorname{im} B)$$

This iteration will also hit a fixed point after at most $\dim(\ker C)$ steps, because $\dim \mathcal{V}_{i+1} \le \dim \mathcal{V}_i$, and

$$(\dim \mathcal{V}_{i+1} = \dim \mathcal{V}_i) \Rightarrow (\mathcal{V}_{i+1} = \mathcal{V}_i)$$

The fixed point $\mathcal{V}^*$ satisfies:

$$\mathcal{V}^* = \mathcal{V}^* \cap A^{-1}(\mathcal{V}^* + \operatorname{im} B)$$

$$\mathcal{V}^* \subset A^{-1}(\mathcal{V}^* + \operatorname{im} B)$$

$$A\mathcal{V}^* \subset \mathcal{V}^* + \operatorname{im} B$$

Thus: $\mathcal{V}^*$ is CI, $\mathcal{V}^* \subset \ker C$

How do we know that $\mathcal{V}^*$ is maximal?

Suppose that $\mathcal{W}$ is CI, and $\mathcal{W} \subset \mathcal{V}_i$ for some i, we can show that $\mathcal{W} \subset \mathcal{V}_{i+1}$: This is because

$\mathcal{V}_{i+1} = \mathcal{V}_i \cap A^{-1}(\mathcal{V}_i + \operatorname{im} B)$, and $\mathcal{W} \subset \mathcal{V}_i$, and

$\mathcal{W} \subset A^{-1}(\mathcal{W} + \operatorname{im} B) \subset A^{-1}(\mathcal{V}_i + \operatorname{im} B)$

Thus $\mathcal{V}^*$ contains all CI spaces in $\ker C$