Consider the state-space representation:
\[ x' = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \]
\[ y = Cx, \quad y \in \mathbb{R}^l \]

A subspace \( V \subset \mathbb{R}^n \) is an invariant subspace (or \( A \)-invariant subspace) if: \( \forall x \in V, \ Ax \in V \)

Short-hand notation: \( AV = V \)

Interpretation: Set the input \( u = 0 \). Any initial state in \( V \) results in a trajectory that remains in \( V \).

Example: Suppose that \( A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \), \( n = 2 \).

The space \( V = \text{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) is invariant. This is because for any vector \( x \in V \),
\[ x = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}, \quad Ax = \begin{bmatrix} -\lambda \\ 0 \end{bmatrix} \in V. \]

Note: You can check invariance by checking the product of \( A \) and a set of basis vectors of \( V \) (from linearity).

Note: Any eigenvector of \( A \) spans a 1-dimensional invariant space of \( A \).

A subspace \( V \subset \mathbb{R}^n \) is a controlled-invariant subspace (or \( (A,B) \)-invariant subspace) if: \( \forall x \in V \), there exists a control vector \( w \in \mathbb{R}^m \) such that \( Ax + Bu \in V \) (\( w \) generally depends on \( x \)).

\[ V + W = \{ z \mid \exists x \in V, \ y \in W, \ z = x + y \} \]
Short-hand notation:
\[ AV \subseteq V + \text{im } B \]

Read: Any element of \( AV \) can be written as a sum of an element of \( V \) and an element of \( \text{im } B \). Thus:

For any \( x \in V \), there exist \( q \in V \) and \( \hat{w} \in \mathbb{R}^m \) such that
\[ Ax = q + \hat{w} \]
\[ Ax - B\hat{w} = q, \text{ on} \]
\[ Ax + B\hat{w} = q', \text{ by renaming } \hat{w} = -\hat{w}. \]

Interpretation: we can always find an input that makes the space \( V \) invariant.

Example: \[ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (n=2, m=1) \]

The space \( V = \text{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is controlled invariant. To see this, take any \( x \in V \), \( x = \begin{bmatrix} k \\ k \end{bmatrix} \) \( \rightarrow \) \( Ax = \begin{bmatrix} -k \\ k \end{bmatrix} \), take \( w = -2k \), then
\[ Ax + Bw = \begin{bmatrix} -k \\ k \end{bmatrix} + \begin{bmatrix} b \\ -2k \end{bmatrix} = \begin{bmatrix} -k \\ -k \end{bmatrix} \in V \]

Note: Again, you can check by taking a set of basis vectors in \( V \).

In fact, this will reveal an interesting relation:

Suppose that \( V \) is a \( p \)-dimensional subspace and \( (x_1, x_2, \ldots, x_p) \) is a set of basis vectors of \( V \).

Suppose that for \( i = 1 \ldots p \), there exist \( w_i \in \mathbb{R}^m \) such that
\[ Ax_i + Bw_i = q_i \in V, \quad i = 1 \ldots p. \]

Any other vector \( x \in V \) can be written as:
\[ x = a_1x_1 + a_2x_2 + \ldots + a_px_p, \quad a_1 \ldots p \in \mathbb{R} \]

Define \( w = a_1w_1 + a_2w_2 + \ldots + a_pw_p \in \mathbb{R}^m \), then
\[ Ax + Bw = A \sum_{i=1}^{p} x_i \sum_{i=1}^{p} w_i + B \sum_{i=1}^{p} x_i w_i \]

\[ = \sum_{i=1}^{p} x_i w_i \in V \]

Notice that \( W \) can be computed as a linear feedback:

\[ W = [w_1 : w_2 : \ldots : w_p] [x_1 : x_2 : \ldots : x_p]^T \]

\[ x = [x_1 : x_2 : \ldots : x_p] [x_1 : x_2 : \ldots : x_p]^T, \text{ thus} \]

\[ W = [w_1 : w_2 : \ldots : w_p] [x_1 : x_2 : \ldots : x_p]^T x \]

\[ W = Fx \]

\[ [x_1 : x_2 : \ldots : x_p]^T \text{ is the left pseudo inverse of } [x_1 : x_2 : \ldots : x_p] \]

\[ X^T = (X^T X)^{-1} X^T \]

This also means that \( V \) is an invariant space of the closed-loop system \( x = (A + BF)x \)

In the previous example:

\[ W = -2k \begin{bmatrix} K \end{bmatrix} [K]^{-1} [K \ k] x \]

\[ = -2k \begin{bmatrix} K & K \end{bmatrix} x = -[1 \ 1] x \]

**Lemma:** If \( V \) and \( W \) are both \((A, B)\) invariant subspaces, then \( V + W \) is also \((A, B)\) invariant.

**Proof:** For any \( x \in V + W \), there exist \( x_1 \in V \) and \( x_2 \in W \) such that \( x = x_1 + x_2 \). However, since \( V \) and \( W \) are both \((A, B)\) invariant, we have \( w_1, w_2 \in R^m \) such that

\[ Ax_1 + Bw_1 \in V; \ Ax_2 + Bw_2 \in W \]
Define \( W = W_1 + W_2 \), then
\[
A_2 x + B_2 w = A x_1 + A x_2 + B w_1 + B w_2 \in V + W
\]

**Disturbance Decoupling Problem (DDP)**

Consider the state space representation with disturbance:
\[
\dot{x} = Ax + Bu + Cd, \quad d \in \mathbb{R}^d
\]
\[
y = Cx
\]

Problem: we want to find a linear feedback control \( u = -Kx \) such that the disturbance \( d \) is perfectly decoupled from the system, i.e. the TF from \( d \) to \( y \) is 0.

Recall that the space \( \{ x \in \mathbb{R}^n \mid CX = 0 \} \) is called \( \ker C \), which is a space of states corresponding to zero output (not to be confused with unobservable space).

**Idea:** Design the linear feedback such that there is a subspace \( V \):
\[
V \subsetneq \ker C \quad \ldots \quad (\dagger)
\]
\[
(A - BK)V \subset V \quad \ldots \quad (\ast)
\]
\[
\text{im } G \subset V \quad \ldots \quad (**)\]

**Theorem:** There exists a unique maximal controlled invariant subspace in \( \ker C \). We denote it by \( V^* \).

**Proof:** There is always a CI space with dimension 0, i.e. the origin. Perform the following procedure: \( V_0 = \{0\} \)

Set \( K = 0 \);

If \( \exists W \subset \ker C \), \( W \subsetneq V_k \), then
\[
\exists V_{k+1} = V_k + W;
\]
\[
k = k + 1;
\]

The procedure above will terminate after at most \( \dim(\ker C) \) steps.
the final value is $V^*$

**Theorem:** DDP is solvable if and only if $\text{im } G \subseteq V^*$

**How to calculate $V^*$?**

Define: $A^{-1}V \triangleq \{ x | A x \in V \}$

Iteration:

$V_0 = \ker C$

$V_{i+1} = V_i \cap A^{-1}(V_i + t \text{im } B)$

This iteration will also hit a fixed point after at most $\dim(\ker C)$ steps, because

$\dim V_{i+1} \leq \dim V_i$, and

$(\dim V_{i+1} = \dim V_i) \Rightarrow (V_{i+1} = V_i)$

The fixed point $V^*$ satisfies:

$V^* = V^* \cap A^{-1}(V^* + t \text{im } B)$

$V^* \subseteq A^{-1}(V^* + t \text{im } B)$

$AV^* \subseteq V^* + t \text{im } B$

Thus: $V^*$ is CI, $V^* \subseteq \ker C$

**How do we know that $V^*$ is maximal?**

Suppose that $W$ is CI, and $W \subseteq V_i$ for some $i$, we can show that $W \subseteq V_{i+1}$: This is because

$V_{i+1} = V_i \cap A^{-1}(V_i + t \text{im } B)$, and $W \subseteq V_i$, and

$W \subseteq A^{-1}(W + t \text{im } B) \subseteq A^{-1}(V_i + t \text{im } B)$

Thus $V^*$ contains all CI spaces in $\ker C$