Some bits about Multivariable H∞ synthesis (Section 3.5.7)

Recall that the transfer matrix from:
- \( r \) to \( e \) is \( (I + 6K)^{-1} = S \)
- \( r \) to \( u \) is \( K(I + 6K)^{-1} = KS \)
- \( n \) to \( y \) is \( -(I + 6K)^{-1}6K = -T \)

Mixed synthesis \( S/KS/T \)

\[
N = \begin{bmatrix}
W_p S \\
W_u K S \\
W_T T
\end{bmatrix}
\]

Goal: \( \|NNu\| \leq 1 \)

The weights can be taken as diagonal matrices, with diagonal entries formulated as in SISO case.

Example: Apply H∞ synthesis for the MIMO problem discussed in the previous lecture, say for closed bandwidth = 100 rad/s.

Formula \( \text{Han} \): \( \|W_p S\| \leq 1 \), with \( W_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

\( \frac{s/M + W_C}{s + W_C A} \)

\( W_C = 100, \ A = 10^{-3}, \ M = 1.05 \)
General Control Problem Formulation

![Block diagram](image)

**General idea:**
- \( w \) and \( z \) are used in characterizing performance
- \( u \) and \( v \) are used in interconnection with controller

**Control problem:** Design \( K \) such that the TF from \( w \) to \( z \) is small.

**Examples:** 1 DOF Feedback Control

**Identify the general variables:**
- \( W = \begin{bmatrix} \mathbf{d} \\ \mathbf{r} \end{bmatrix} \);
- \( Z = Y - r \)

- \( U = U \);
- \( V = e = r - y - n \)

**The generalized plant model consists of a TF from \( [w] \) to \( [z] \)**

**To compute this, cut the control loop.**

- \( Z = y - r \)
- \( = G u + d - r \)
- \( V = r - Gu - d - n \)

Thus:
\[
W = \begin{bmatrix} I & -I & 0 & G \\ -I & I & -I & -G \end{bmatrix} \begin{bmatrix} d \\ r \\ n \\ u \end{bmatrix}
\]
Example: Disturbance feedforward control

Identify the general variables:

\[ w = \begin{bmatrix} q \\ r \\ n \end{bmatrix}, \quad z = y - r \]

\[ u = u, \quad v = \begin{bmatrix} r - y - n \\ d \end{bmatrix} \]

\[ z = y - r = 6u + d - r \]

\[ v = \begin{bmatrix} r - y - n \\ d \end{bmatrix} = \begin{bmatrix} r - 6u - d - n \\ d \end{bmatrix} \]

The transfer matrix:

\[ \begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} I & -I & 0 & 0 & G \\ -I & I & -I & -G & 0 \\ I & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \]

Partitioning \( P \): We often partition \( P \) as follows:

\[ z = p_{11} w + p_{12} u \]

\[ v = p_{21} w + p_{22} u \]

Once we close the loop with \( u = K v \), the TF from \( w \) to \( z \) is

\[ z = p_{11} w + p_{12} K v \]

\[ v = p_{21} w + p_{22} K v \quad \Rightarrow \quad v = (I - p_{22} K)^{-1} p_{22} w \]

\[ 2 = \begin{bmatrix} p_{11} + p_{12} K \left( I - p_{22} K \right)^{-1} p_{21} \end{bmatrix} w \]

\[ N = F_2(P, K) \]

\( F_2(P, K) \) is called the lower Linear Fractional Transform (LFT) of \( P \) with \( K \) as the parameter. Controller synthesis amount to finding \( K \) that
Example: S/T/Ks mixed Hoo synthesis

\[ r \xrightarrow{e} K \xrightarrow{u} G \xrightarrow{y} \text{Find } K \text{ that minimizes } ||N||_2, \]
where \( N = \begin{bmatrix} W_p S \\ W_u K S \\ W_T T \end{bmatrix} \)

Recall that:
- \( S \) is the TF from \( r \) to \( e \)
- \( K S \) is the TF from \( r \) to \( u \)
- \( T \) is the TF from \( r \) to \( y \)

Thus, we have the general variables:
- \( W = r \)
- \( e = \begin{bmatrix} W_p \cdot e \\ W_u \cdot u \\ W_T \cdot y \end{bmatrix} \)

The generalized plant TF:
- \( z = \begin{bmatrix} W_p (r - Gu) \\ W_u \cdot u \\ W_T \cdot y \end{bmatrix} \), \( v = e = r - Gu \)

\[
P_{11} = \begin{bmatrix} W_p \\ 0 \\ 0 \end{bmatrix}; \quad P_{12} = \begin{bmatrix} -W_p G \\ W_u \\ W_T G \end{bmatrix}; \quad P_{21} = I; \quad P_{22} = -G
\]

Let's verify with LFT formulas:
\[
N = \begin{bmatrix} W_p \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -W_p G \\ W_u \\ W_T G \end{bmatrix} K (I + GK)^{-1} I
\]

\[
= \begin{bmatrix} W_p - W_p T \\ W_u K S \\ W_T T \end{bmatrix} = \begin{bmatrix} W_p S \\ W_u K S \\ W_T T \end{bmatrix}
\]
In MATLAB, given $P$, $K$ can be synthesized using "hinfsyn".

Note: Not all stacked TF can be represented as LFT. Read Schönh 3.8.7

**Including Model Uncertainty**

**General idea:** The plant model might have uncertainties due to various factors such as parameter uncertainty, unmodeled dynamics, etc.

The philosophy of robust controller synthesis is to design controllers for which the plant is stabilized regardless of uncertainty. This will be discussed later.

The way to include the uncertainty in the generalized plant model:

![Plant and Controller Diagram](image)

Here $\Delta$ is a diagonal transformation matrix capturing the individual $\Delta_i$ blocks.

$\Delta_i$ are typically normalized such that $\|\Delta_i\|_\infty \leq 1$. Once we close the control loop, assume that we have a TF:

\[
\begin{bmatrix} y \Delta \\ z \end{bmatrix} = \begin{bmatrix} N_{11} U \Delta + N_{12} W \\ N_{21} U \Delta + N_{22} W \end{bmatrix}
\]

Given $\Delta$, the TF from $z$ to $W$ can be found through the Upper Fracturc Transformation:

\[
F_u(N, \Delta) = N_{22} + N_{21} \Delta (I - N_{11} \Delta)^{-1} N_{12}
\]