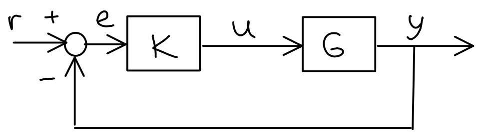
Performance Limitations

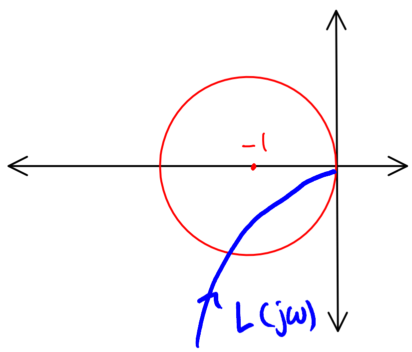


$L = GK$

Consider SISO system: $\quad S = \dfrac{1}{1+L}$

We have learned that the peak of $|S|$ corresponds to tracking performance. We have also observed that in most cases, the peak of $|S|$ is larger than 1. Is there a general rule about this?

$|S| = \dfrac{1}{|1+L|}$, thus $|S| > 1$, if $L(s)$ enters the unit circle around -1



Most physical systems are strictly proper because generally, excitation at high frequency range requires more energy, i.e the frequency response goes to zero as $s \to j\infty$. Since we assume proper controller, then the loop TF L is strictly proper too, or

$$\lim_{\omega \to \infty} |L(j\omega)| = 0$$

If the pole excess is two or more, and there is no RHP zero, $L(j\omega)$ will enter the circle, thus the peak of $|S| > 1$ is unavoidable

Bode sensitivity Integral (Waterbed formula): If the pole excess is two or more and there is no RHP zero, then

$$\int_0^\infty \ln |S(j\omega)| \, d\omega = \pi \cdot \sum_{i=1}^{N_p} \mathrm{Re}(p_i)$$

where $p_{1,2,\dots,N_p}$ are RHP poles

Interpretation: $\ln|S| > 0 \Leftrightarrow |S| > 1$. Thus unstability of OL plant contributes to the sensitivity.

* The constant integral implies "conservation of sensitivity", hence the waterbed analogy

Systems with one real RHP zero or a pair of complex conjugate RHP zeros

Theorem: If L has one RHP zero at $s = z$, or a pair of complex conjugate RHP zeros at $s = z = x \pm jy$

$$\int_0^\infty \ln|S(j\omega)|\, W(z,\omega)\, d\omega = \pi \cdot \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z}{p_i - z} \right|$$

where

$$W(z,\omega) = \frac{2z}{z^2 + \omega^2} \quad \text{if } z \text{ is Real}$$

$$W(z,\omega) = \frac{x}{x^2 + (y-\omega)^2} + \frac{x}{x^2 + (y+\omega)^2}$$

Interpretation: as $\omega \to \infty$, $W(z,\omega) \to 0$, thus the contribution of frequencies larger than ω to the integral diminish. This practically limits the attainable CL bandwidth at around z

Bounds on Peak of S

Thm: For closed loop stability, for any RHP zero z of $G(s)$, the following holds

$$\|W_p S\|_\infty \geq |W_p(z)| \cdot \prod_{i=1}^{N_p} \left| \frac{z + p_i}{z - p_i} \right|$$

$p_i, i = 1, \dots, N_p$ are RHP poles of $G(s)$.

If there is no RHP poles, then $\|W_p S\|_\infty \geq |W_p(z)|$

Special case: without weight $\to W_p(s) = 1$,

$$M_S \triangleq \|S\|_\infty \geq \prod_{i=1}^{N_p} \left| \frac{z + \bar{p}_i}{z - p_i} \right|$$

Remark:

- For systems with one RHP zero, no time delay, this bound is tight

Proof: Because of internal stability, it is not desirable to have RHP pole-zero cancellation. Therefore, for any RHP zero of $G(s)$, z,

$$G(z) = 0, \quad L(z) = 0, \quad S(z) = \frac{1}{1+0} = 1.$$

Thus: $\|W_p S\|_\infty = \max_\omega |W_p(j\omega) S(j\omega)| \geq \max_{s \in \mathbb{C}_+} |W_p(s) S(s)| \geq |W_p(z)|$

Obviously, when there are RHP poles of $G(s)$, this bound still holds. However, in this case, we can construct a tighter bound.

For any RHP pole of $G(s)$, p,

$$G(p) = \infty, \quad L(p) = \infty, \quad S(p) = \frac{1}{1 + L(p)} = 0$$

Thus p is an RHP zero of S.

We can split $S(s)$ into an allpass filter $S_a(s)$ that contains all RHP zeros of S, and $S_m(s)$:

$$S = S_m \cdot S_a, \text{ where } S_a(s) = \prod_{i=1}^{N_p} \frac{s - p_i}{s + \bar{p}_i}$$

Example: If $S = \frac{s-1}{s+2}$, then $S = \underbrace{\frac{s+1}{s+2}}_{S_m} \cdot \underbrace{\frac{s-1}{s+1}}_{S_a}$

Observe that if $p_i = a + jb$, then:

$$\frac{j\omega - p_i}{j\omega + \bar{p}_i} = \frac{j\omega - a - jb}{j\omega + a - jb} = \frac{-a + j(\omega - b)}{a + j(\omega - b)},$$

thus: $\left|\frac{j\omega - p_i}{j\omega + \bar{p}_i}\right| = 1$, and therefore $|S_a(j\omega)| = 1$

We then have:

$$\|W_p S\|_\infty = \max_\omega |W_p(j\omega) S(j\omega)| = \max_\omega |W_p(j\omega) S_m(j\omega)|$$

$$\geq |W_p(z)| \, |S_m(z)|$$

But $|S_m(z)| = \frac{1}{|S_a(z)|} = \prod_{i=1}^{N_p} \left|\frac{z + \bar{p}_i}{z - p_i}\right|$

Example: If $G(s) = \frac{s-1}{s-2}$, then $\|S\|_\infty \geq \frac{3}{1} = 3$

Bounds on Peak of T

Thm: For any RHP pole of $G(s)$, p,

$$\|W_T T\|_\infty \geq |W_T(p)| \cdot \prod_{j=1}^{N_z} \left|\frac{\bar{z}_j + p}{z_j - p}\right| \cdot |e^{p\theta}|$$

where $z_i, i = 1, \ldots, N_z$ are the RHP zeros of $G(s)$ and θ is the time delay of $G(s)$.

If $G(s)$ does not have any RHP zero or time delay,

$$\|W_T T\|_\infty \geq |W_T(p)|$$

Without weight, i.e $W_T = 1$,

$$\|T\|_\infty \geq \prod_{j=1}^{N_z} \left|\frac{\bar{z}_j + p}{z_j - p}\right| \cdot |e^{p\theta}|$$

Note: For systems with one RHP pole the bound is tight.

Examples: $G(s) = \dfrac{e^{-0.5s}}{s-3}$ $\leftarrow$ 0.5 sec time delay ($\theta = 0.5$)
$\leftarrow$ RHP pole ($p = 3$)

$$\|T\|_\infty \geq |e^{p\theta}| = e^{3 \cdot 0.5} \approx 4.48$$

A bound on $\|S\|_\infty$ can be found from $|S| \geq |T| - 1$, thus

$$\|S\|_\infty \geq 3.48 \quad \text{(not necessarily tight)}$$

Bounds on the peak of SG

Important for limiting the influence of input disturbance

To obtain a bound, we define the stable and minimum phase version of G:

$$G = G_{ms} \cdot \prod \frac{s - z_i}{s + z_i} \cdot \prod \frac{s + p_i}{s - p_i}$$

where z_i and p_i are RHP zeros and poles of G. We have

$$|G| = |G_{ms}|, \text{ thus}$$

$\|SG\|_\infty = \|SG_{ms}\|_\infty$. From here, we treat G_{ms} as a weighting function W_p, and obtain a bound

$$\|SG\|_\infty \geq |G_{ms}(z)| \prod_{i=1}^{N_p} \left| \frac{z + p_i}{z - p_i} \right|, \text{ for any } z \text{ RHP zero of } G$$

This bound is tight for systems with single RHP zero.