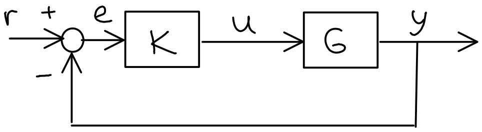
Performance Limitations for MIMO systems



$L = GK$

Sensitivity Integral

Suppose that $L(s)$ has entries with pole excess two or more, then the sensitivity function $S = (I+GK)^{-1}$ satisfies

$$\int_0^\infty \ln|\det S(j\omega)|\, d\omega = \pi \cdot \sum_{i=1}^{N_p} \mathrm{Re}(p_i),$$

where $p_i, i=1$

How does $\det S(j\omega)$ correspond $\sigma_{max} S(j\omega)$? Through SVD, we can write

$S(j\omega) = U \Sigma V^H$, where U and V are unitarian matrices

Σ is the diagonal matrix of singular values:

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ & & \ddots \\ 0 & & \sigma_n \end{bmatrix}$$

$$|\det S(j\omega)| = |\det(U)||\det(\Sigma)| \cdot |\det(V)| = \prod_{i=1}^{n} \sigma_i$$

Therefore: $\ln|\det S(j\omega)| = \sum_{i=1}^{n} \ln \sigma_i S(j\omega) \leq \frac{1}{n} \ln \sigma_{max} S(j\omega)$

Review of MIMO pole/zero output direction

Consider an $l \times m$ MIMO plant $G(s)$. The normal rank of $G(s)$ is

$$r = \min(l, m)$$

Using SVD, we can decompose

$$G = U \Sigma V^H$$, all depend on s

$z \in \mathbb{C}$ is a zero of $G(s)$ if $\Sigma(s)$ loses rank at $s = z$ (Normally the rank of Σ is r)

Example: $G(s) = \dfrac{1}{(s+2)(s+1)} \begin{bmatrix} s & -1 \\ 100 & s \end{bmatrix}$, $r = 2$

$G(s)$ loses rank if $\det G(s) = 0 \Rightarrow \dfrac{s^2+100}{(s+2)^2(s+1)^2} = 0$, $s = \pm j10$

For zero at $s = j10$: $G(j10) = \dfrac{1}{-8+j30} \begin{bmatrix} j10 & -1 \\ 100 & j10 \end{bmatrix}$

$G(j10) = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$, $\begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix}$ is the output direction of the zero at $s = j \cdot 10$

Similar calculation can be performed for the other zero.

$p \in \mathbb{C}$ is a pole of $G(s)$ if at $s = p$, some entries of Σ becomes infinite.

Pole polynomial: the least common denominator of all minors of $G(s)$.

Example: $G(s) = \dfrac{1}{s+2} \begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$

Minors of order 1: $\dfrac{s-1}{s+2}$, $\dfrac{4}{s+2}$, $\dfrac{4.5}{s+2}$, $\dfrac{2(s-1)}{s+2}$

Minor of order 2: $\dfrac{2(s-1)^2 - 4 \cdot 4.5}{(s+2)^2} = \dfrac{2s^2 - 4s + 2 - 18}{(s+2)^2} = \dfrac{2(s-4)\cancel{(s+2)}}{(s+2)^{\cancel{2}}}$

Thus the polynomial is $(s+2)$. There is one pole at $s = -2$

To see the output direction of the pole, substitute in $s = -2$ to $G(s)$

$$G(-2) = \frac{1}{-2+2} \underbrace{\begin{bmatrix} -3 & 4 \\ 4.5 & -6 \end{bmatrix}}_{\text{rank} = 1}$$

Thus the pole's output direction is the unit vector of $\begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{13} \\ -3/\sqrt{13} \end{bmatrix}$

Example: $G(s) = \dfrac{1}{(s+2)(s+1)} \begin{bmatrix} s & -1 \\ 100 & s \end{bmatrix}$

Minors of order 1 have $(s+2)(s+1)$ as denominator

Minors of order 2: $\dfrac{s^2+100}{(s+2)^2(s+1)^2}$

Thus the pole polynomial is $= (s+2)^2(s+1)^2$

For two poles at $s=-1$, we see that

$$G(-1) = \frac{1}{1 \cdot 0} \underbrace{\begin{bmatrix} -1 & -1 \\ 100 & -1 \end{bmatrix}}_{\text{rank } 2}$$

The two pole output directions can be taken as, e.g. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Similarly for the two poles at $s=-2$.

From state space representation:

Suppose that $\dot{x} = Ax + Bu$

$y = Cx + Du$ is a minimal representation.

Recall that the poles are the eigenvalues of A. The poles' state directions are the corresponding eigenvectors. The output directions are $C \cdot$ state directions

bounds on Peaks

Thm: Given a plant with no delay, $G(s)$. Suppose that $G(s)$ has z_i, $i = 1 \dots N_z$ as RHP zeros with output directions $y_{z,i}$, $i = 1 \dots N_z$. Also, suppose that p_i, $i = 1 \dots N_p$ are RHP poles with output directions $y_{p,i}$, $i = 1 \dots N_p$. Also suppose that all RHP poles and zeros are distinct,

$$\|S\|_\infty \geq \Lambda$$

$$\|T\|_\infty \geq \Lambda, \text{ where } \Lambda = \sqrt{1 + \bar{\sigma}^2\left(Q_z^{-1/2} Q_{zp} Q_p^{-1/2}\right)}$$

where $Q_z \in \mathbb{C}^{N_z \times N_z}$, $[Q_z]_{ij} = \dfrac{y_{z,i}^H y_{z,j}}{z_i + \bar{z}_j}$

$$Q_p \in \mathbb{C}^{N_p \times N_p}, \quad [Q_p]_{ij} = \frac{y_{p,i}^H y_{p,j}}{\bar{p}_i + p_j}$$

$$Q_{zp} \in \mathbb{C}^{N_z \times N_p}, \quad [Q_{zp}]_{ij} = \frac{y_{z,i}^H y_{p,j}}{z_i - p_j}$$

Example: SISO case $\quad G(s) = \dfrac{(s-1)(s-3)}{(s-2)(s+1)^2}$

Two RHP zeros +1 and +3, one RHP pole at +2. All pole and zero output directions can be taken as 1, thus

$$Q_z = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} ; \quad Q_p = \frac{1}{4} ; \quad Q_{zp} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$Q_z^{-1/2} Q_{zp} Q_p^{-1/2} = \begin{bmatrix} -7.953 \\ 12.679 \end{bmatrix}$$

$$\Lambda = \sqrt{1 + \left\| \begin{matrix} -7.953 \\ 12.679 \end{matrix} \right\|^2} \approx 15$$

compare with SISO bound from last week.

Special case: One RHP zero and pole:

$$\Lambda = \sqrt{\sin^2\phi + \frac{|z+p|^2}{|z-p|^2}\cos^2\phi}$$

, where ϕ is the angle between the pole and zero directions.