Robust Stability

Question: How to ensure that a controller designed for nominal plant ($\Delta = 0$) also stabilizes any perturbed plant?

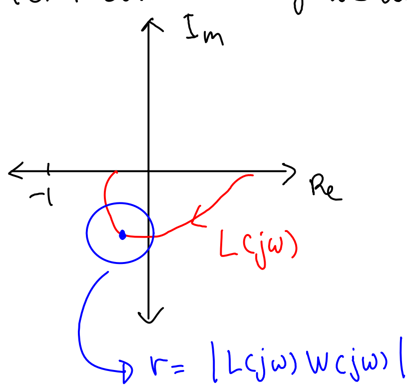
Suppose that the plant uncertainty is modeled as multiplicative uncertainty
$G_p = G(1 + W\Delta)$, $\|\Delta\|_\infty \le 1$, and the controller is K.

Assume that G does not have RHP pole (can be relaxed later), and the CL system with controller K is stable. Then, the Nyquist plot of $L = GK$ does not encircle -1.

For any perturbed plant, the loop gain

$$\begin{aligned} L_p = G_p K &= G(1 + W\Delta)K \\ &= GK + GKW\Delta \\ &= L + LW\Delta \end{aligned}$$

For robust stability we want $L_p(j\omega)$ not to encircle -1, for any Δ.



Necessary and sufficient condition for robust stability:

$$|L(j\omega) W(j\omega)| < |-1 - L(j\omega)| = |1 + L(j\omega)|$$

or

$$\left| \frac{L(j\omega)}{1 + L(j\omega)} \right| |W(j\omega)| < 1$$

or equivalently

$$\|TW\|_\infty < 1$$

Actually the same condition applies for unstable G as well, because we basically require that the number of encirclements does not change

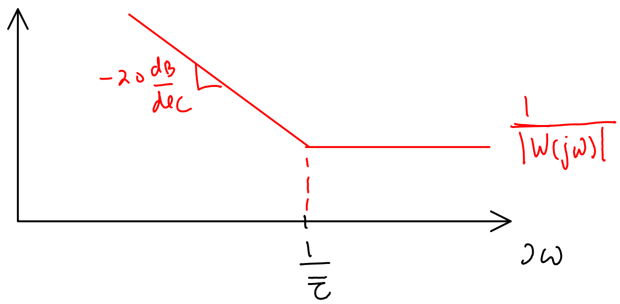
Note that if Δ is constraint to be real, then this condition is only sufficient

Recall the example with a family of uncertain plant models:

$$G_p(s) = (1+s\tau)\,G_0(s) \;;\; \tau_{min} \le \tau \le \tau_{max}$$

We have computed that: $G_p(s) = G(s)(1+W(s)\Delta)$, where

$$W(s) = \frac{1}{2}\,\frac{s(\tau_{max}-\tau_{min})}{1+s\bar{\tau}}, \quad \text{where } \bar{\tau} = \frac{\tau_{min}+\tau_{max}}{2}$$



Inverse Multiplicative Uncertainty

In this case the uncertainty is expressed as:

$$G_p = G(1+W\Delta)^{-1}, \quad \|\Delta\|_\infty \le 1$$

With a controller K,

$$L_p = G(1+W\Delta)^{-1}K = \frac{GK}{1+W\Delta} = \frac{L}{1+W\Delta}$$

For robust stability, a necessary and sufficient condition

$$|1+L_p| > 0 \iff \left|\frac{1+W\Delta+L}{1+W\Delta}\right| > 0 \iff |1+W\Delta+L| > 0$$

We can show that:

$\exists \Delta \in \mathbb{C}$, $|\Delta| \le 1$ such that $1+W\Delta+L=0$ if and only if

$$|W| \ge |1+L|$$

Thus, a necessary and sufficient condition for robust stability:

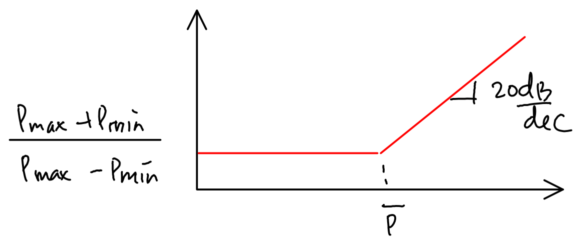
$$|W| < |1+L|, \text{ or equivalently}$$

$$\left|\frac{1}{1+L}\right||W| < 1, \text{ or equivalently}$$

$$\|SW\|_\infty < 1$$

Recall the example: $G_p(s) = \dfrac{G_0(s)}{s+p}$, $p_{min} \le p \le p_{max}$

In this case, $W(s) = \dfrac{1}{2}\dfrac{p_{max} - p_{min}}{s + \bar{p}}$, where $\bar{p} = \dfrac{p_{max} + p_{min}}{2}$

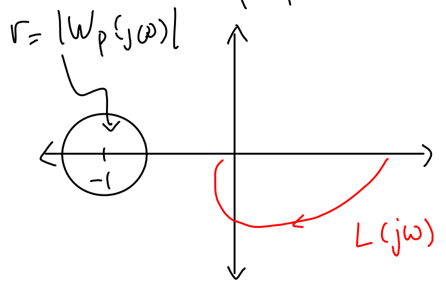


Robust Performance

Question: How to ensure that a controller K that is designed for the nominal plant G such that

$$\|W_p S\|_\infty = \left\|W_p \frac{1}{1+GK}\right\|_\infty \le 1 \quad \text{(nominal performance)}$$

still meets the performance criterion with perturbed plant.



Note that: $\|W_p S\|_\infty \Leftrightarrow \exists \omega$,
$|W_p(j\omega)| < |1 + L(j\omega)|$

For multiplicative uncertainty: $L_p = L(1 + W\Delta)$, $\|\Delta\|_\infty \le 1$

A necessary and sufficient condition for robust performance is thus:

$$\forall \omega, \quad |W_p(j\omega)| < |1 + L_p(j\omega)| = |1 + L(j\omega) + L(j\omega)W(j\omega)\Delta(j\omega)|$$

We can show that $\exists\, \Delta \in \mathbb{C}$, $|\Delta| \le 1$ such that

$$|W_p(j\omega)| \ge |1 + L(j\omega) + L(j\omega)W(j\omega)\Delta(j\omega)| \text{ iff}$$
$$|1 + L(j\omega)| - |W_p(j\omega)| \le |L(j\omega)W(j\omega)|$$

Thus a necessary and sufficient condition for robust performance is:

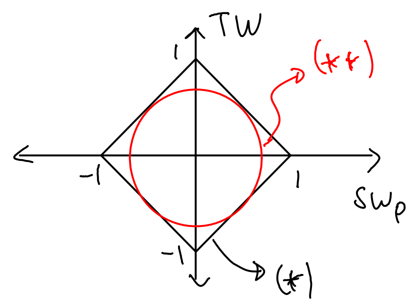
$$|L(j\omega)W(j\omega)| + |W_p(j\omega)| < |1+L(j\omega)| \quad \text{, or}$$

$$\boxed{|T(j\omega)W(j\omega)| + |S(j\omega)W_p(j\omega)| < 1} \quad , \forall \omega \qquad (*)$$

Note that this cannot be expressed as H_∞ norm bound, however we can (conservatively) approximate it by a mixed S/T synthesis:

$$\left\| \begin{matrix} SW_p \\ TW \end{matrix} \right\|_\infty \leq \frac{1}{2}\sqrt{2} \quad (**)$$

Illustration:



Recap: (Nominal Performance) $|SW_p| < 1$

(Robust Stability) $|TW| < 1$

(Robust Performance) $|SW_p| + |TW| < 1$

Therefore RP implies both NP and RS.

What about inverse multiplicative uncertainty?

We have $L_p = \frac{L}{1+W\Delta}$, $\|\Delta\|_\infty \leq 1$

If nominal performance is given by $|TW_T| < 1$, then as an N&S condition for RP we have:

$$\forall \omega \quad |W_T||L_p| < |1+L_p|$$

$$\Leftrightarrow \frac{|W_T||L|}{|1+W\Delta|} < \left| \frac{1+W\Delta+L}{1+W\Delta} \right|$$

$$\Leftrightarrow |W_T||L| < |1+W\Delta+L|$$

$$\Leftrightarrow |W| < |1+L| - |W_T||L| \Leftrightarrow \boxed{|SW| + |TW_T| < 1}$$