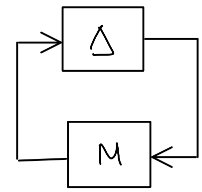
Robust Stability of the MΔ structure

Recall that RS can be expressed as the stability of the loop:



This loop is stable iff the poles of $(I - M\Delta)^{-1}$ are stable, or equivalently iff the zeros of $(I - M\Delta)$ are stable (OLHP). Notice that the zeros of $(I - M\Delta)$ are the zeros of $\det(I - M(s)\Delta(s))$

Nominal stability $\Rightarrow M(s)$ is stable. We also assume that $\Delta(s)$ is stable. Thus, $\det(I - M(s)\Delta(s))$ does not have any RHP pole.

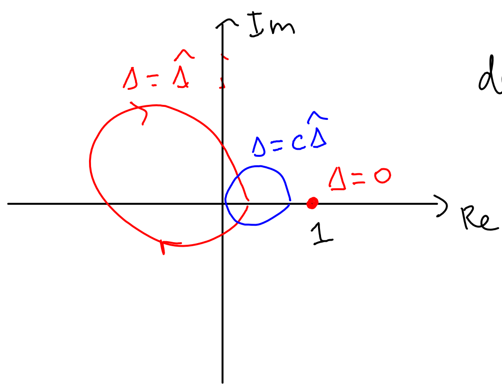
Apply Nyquist criterion: $(I - M\Delta)^{-1}$ is stable iff $\det(I - M(j\omega)\Delta(j\omega))$ does not encircle 0.

Thm: Assuming that $M(s)$ and $\Delta(s)$ are stable, and that if $\Delta(s)$ is an allowed perturbation, then $c\Delta(s)$ is also an allowed perturbation for any real number $c \in [0,1]$, we have

$$\text{Robust stability} \Leftrightarrow \det(I - M(j\omega)\Delta(j\omega)) \neq 0, \ \forall \omega, \forall \Delta$$

Proof: $\Rightarrow$ trivial

$\Leftarrow$ by contradiction. Suppose that not RS, then there exists a $\hat{\Delta}(s)$ such that the origin is encircled. We can then use the factor c to shrink the contour such that it passes through 0.



Note that:

$$\det(I - M\Delta) = 0 \Leftrightarrow 1 \text{ is an eigenvalue of } M\Delta$$

Therefore:

$$\boxed{\text{RS iff } 1 \text{ is not an eigenvalue of } M\Delta \text{ for any } \omega, \text{ and any } \Delta.}$$

This theorem can be strengthened as follows:

Theorem: Assuming that $M(s)$ and $\Delta(s)$ are stable, and that if $\Delta(s)$ is an allowed perturbation, then $C\Delta(s)$ is also an allowed perturbation for any complex C, $|C| \leq 1$, we then have:

$$RS \Longleftrightarrow \rho(M\Delta) < 1, \ \forall \omega, \forall \Delta$$

Note: $\rho(X)$ is $\max \{|\lambda_i|, \lambda_i \text{ is an eigenvalue of } X\}$

Proof: Similar as before, we need to show that:

$$\forall \omega, \forall \Delta \ \det(I - M\Delta) = 0 \Longleftrightarrow \forall \omega, \forall \Delta, \ \rho(M\Delta) < 1$$

($\Longrightarrow$) By contradiction: suppose that $\exists \hat{\omega}$ and $\hat{\Delta}$ such that

$$\rho(M(j\hat{\omega})\hat{\Delta}(j\hat{\omega})) = \hat{\lambda}, \ |\hat{\lambda}| \geq 1,$$

then define $\tilde{\Delta} = \frac{1}{\hat{\lambda}} \hat{\Delta}$. It follows that 1 is an eigenvalue of $M\tilde{\Delta}$, and therefore:

$$\det(I - M(j\hat{\omega})\tilde{\Delta}(j\hat{\omega})) = 0$$

($\Longleftarrow$) By contradiction: suppose that $\exists \hat{\omega}$ and $\hat{\Delta}$ such that

$$\det(I - M(j\hat{\omega})\hat{\Delta}(j\hat{\omega})) = 0, \text{ then}$$

1 is an eigenvalue of $M(j\hat{\omega})\hat{\Delta}(j\hat{\omega})$, and thus $\rho(M(j\hat{\omega})\hat{\Delta}(j\hat{\omega})) \geq 1$.

Robust Stability for Complex Unstructured Uncertainty

Suppose that $\Delta(s)$ is allowed to be any stable TF satisfying $\|\Delta\|_\infty \leq 1$, or equivalently: $\forall \omega$, $\sigma_{max}(\Delta(j\omega)) \leq 1$, then

$$\max_{\Delta} \rho(M(j\omega)\Delta(j\omega)) = \sigma_{max}(M(j\omega))$$

Proof: Perform SVD on $M(j\omega)$ and obtain: $M = U\Sigma V^H$.

Define $\Delta = VU^H$, and then:

$$M\Delta = U\Sigma U^H = U\Sigma U^{-1}$$

Thus $\rho(M\Delta) = \sigma_{max}(M)$. This shows that

$$\max_{\Delta} \rho(M(j\omega)\Delta(j\omega)) \geq \sigma_{max}(M(j\omega))$$

To show the opposite: $\left(\max_{\Delta} \rho(M(j\omega)\Delta(j\omega)) \leq \bar{\sigma}_{max}(M(j\omega))\right) \quad \ldots (*)$

Suppose that λ is an eigenvalue of $M\Delta$, then $\exists\, x$ such that:

$$M\Delta x = \lambda x$$

However: $\|M\Delta x\| \leq \bar{\sigma}_{max}(M) \cdot \bar{\sigma}_{max}(\Delta) \cdot \|x\|$

$$\leq \bar{\sigma}_{max}(M) \cdot \|x\|$$

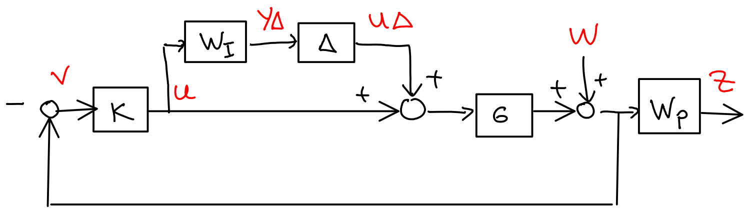
Thus: $|\lambda|\,\|x\| \leq \bar{\sigma}_{max}(M)\,\|x\|$, and therefore $(*)$

We therefore obtain the following necessary and sufficient condition for RS if $\Delta(s)$ is unstructured.

Theorem: Assuming $M(s)$ and $\Delta(s)$ are stable, $\Delta(s)$ is unstructured, we have

$$\boxed{RS \iff \|M\|_\infty < 1}$$

Example: From previous lecture:

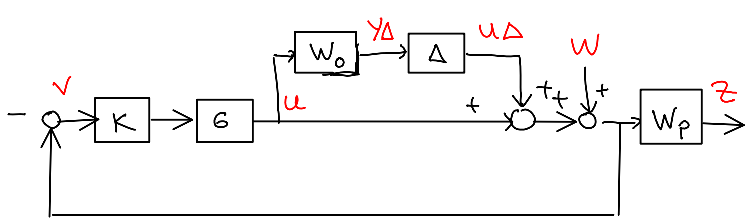


$M = N_{11} = W_I KG(I+KG)^{-1}$, thus we have RS iff

$$\|W_I KG(I+KG)^{-1}\|_\infty < 1$$

Note: $KG(I+KG)^{-1} = \underbrace{(I+KG)^{-1}KG}_{T_I} \neq \underbrace{(I+GK)^{-1}GK}_{T}$

Another example:



M is the TF from u_Δ to y_Δ, thus:

$$M = W_o (I + GK)^{-1} GK = W_o T \quad \text{(apply feedback rule)}$$

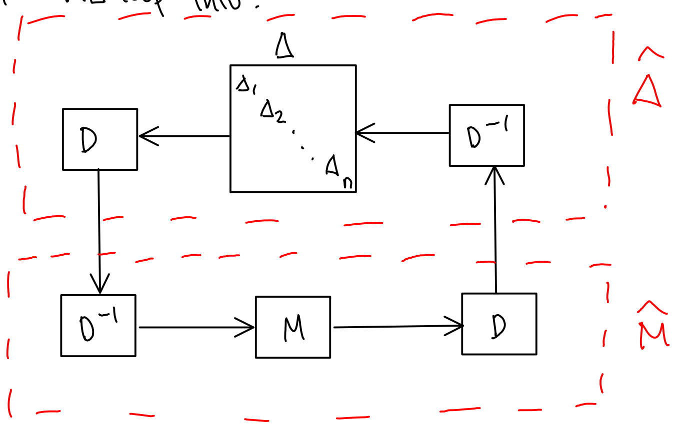
Thus RS iff $\|W_o T\|_\infty < 1$

See section 8.6.1 for more examples

Robust Stability with Structured Uncertainty

Intuition: If Δ is constrained with a structure, the necessary and sufficient condition should be weaker. Why?

Now, convert the $M\Delta$ loop into:



The stability of the $\hat{M}\hat{\Delta}$ loop is the same as that of the $M\Delta$ loop. However, notice that:

$$\hat{\Delta} = D \Delta D^{-1}$$

If D is compatible with the structure of Δ, such that $D\Delta = \Delta D$, then

$$\hat{\Delta} = \Delta \cdot D D^{-1} = \Delta$$, and therefore

M is RS with structured Δ iff $\hat{M}$ is RS with structured Δ.

$\hat{M} = D M D^{-1}$

Therefore, for structured Δ, we obtain the following sufficient condition for RS:

$$RS \Leftarrow \bar{\sigma}_{max}\left(D(\omega) M(j\omega) D^{-1}(\omega) \right) < 1 \quad \forall \omega$$, for some $D(\omega)$ that has the same structure as Δ.