Example (cont'd): (see attached plot figures)

$$G(s) = \frac{200}{10s+1}\,\frac{1}{(0.05s+1)^2}, \quad G_d(s) = \frac{100}{10s+1}$$

with additional requirement:

Disturbance rejection: when $d(t) = 1(t)$, $y(t)$ must be in $[-1, 1]$ all the time, $\lim_{t\to\infty} y(t) = 0$, and for $t > 3s$, $|y(t)| < 0.1$

First design (inverse model based): $K_0(s) = \frac{10}{s}\,\frac{10s+1}{200}\,\frac{0.1s+1}{0.01s+1}$

results in inadequate disturbance rejection (see Fig 1). Upon inspection it is clear that $|K_0(s)| > |G^{-1}(s)G_d(s)|$ only for a very limited bandwidth (see Fig 2)

Controller design for disturbance rejection:

When $r = 0$, $y = SG_d \cdot d \to Y(s) = S \cdot G_d \cdot \frac{1}{s}$; $\lim_{t\to\infty} y(t) = \lim_{s\to 0} S(s)G_d(s)$

$S(s) = \frac{1}{1+L(s)}$ needs to have a zero at $s=0$, $L(s)$ must have a pole at $s=0$.

Idea: $|K| = k \cdot \left|\frac{s+\omega_I}{s}\right| \cdot |G^{-1}G_d|$; Pick $\omega_I \approx 0.2 \cdot \omega_c$

Notice that for $\omega < 20$, $G^{-1}G_d \approx 0.5$, thus try $K_1(s) = \frac{s+2}{s}$

In Fig 2, see that the effective bandwith of $K_1(s)$ is wide enough, and in Fig 3, observe that the step disturbance response already satisfies the requirement.

However, notice in Fig 4 that the step reference response is really poor with overshoot $\sim 80\%$. To address the issue, consider the plot of $T(s)$ and the stability margins in Fig 5 & 6.

Idea: improve stability margins by introducing an additional term

$$\frac{0.1s+1}{0.02s+1} = 5\,\frac{(s+10)}{(s+50)}$$

to bump up the phase response around ω_c.

Thus, define $K_2(s) = \frac{5(s+2)(s+10)}{s(s+50)}$

In Fig 7, we can see that indeed the stability margins improve. In Fig 3-5 we can see some improvement in transient response and peak criterion. However the tracking performance criterion is still not met, since the overshoot is still around 40%

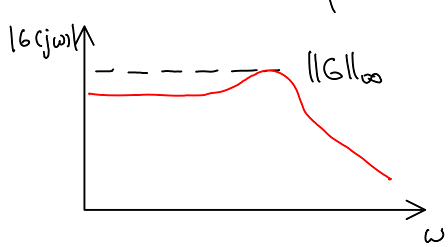
We want to design the prefilter K_r such that:

$$y = T K_r r$$

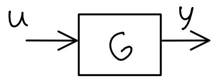
We design a prefilter K_r to level off the peak of $T(s)$ (at around $12\frac{rad}{s}$)
We pick $K_r = 0.1 \cdot \frac{s+90}{s+9}$. See the freq response in Fig 8.
The effects of adding this prefilter can be seen in Fig 9 and 10.
We can see that now both tracking and disturbance rejection criteria are met.

H_∞ control

The term H_∞ refers to the peak magnitude of the freq. response.



It has the interpretation for being the l_2 induced norm of the system.



$$; \quad \|u\|_{\ell_2} = \left[\int_0^\infty |u(t)|^2 dt\right]^{1/2}$$

$$\|y\|_{\ell_2} = \left[\int_0^\infty |y(t)|^2 dt\right]^{1/2}$$

Through Parseval relation:

$$\|u\|_{\ell_2} = \left[\int_0^\infty |u(j\omega)|^2 d\omega\right]^{1/2}$$, similarly with $\|y\|_{\ell_2}$

Suppose that M is the ℓ_2-induced norm of the system, then

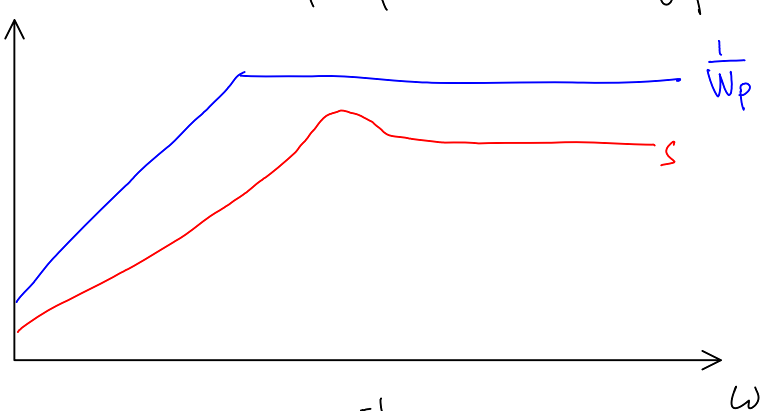
$$M \triangleq \sup_{\|u\|_{\ell_2}=1} \|y\|_{\ell_2}$$, but we also know that

$y(j\omega) = G(j\omega)\,u(j\omega)$. Intuitively, the amplification is the greatest if we allocate the freq content of $u(j\omega)$ around the peak of $G(j\omega)$, thus: $M = \sup_\omega |G(j\omega)| = \|G\|_\infty$

There is an intuitive link between $\|S\|_\infty$ and the total variation (TV)

Shaping the sensitivity functions:

We can bound the shape of the sensitivity function



$$|S(j\omega)| \leq |W_p(j\omega)|^{-1} \rightarrow |S(j\omega)\,W_p(j\omega)| \leq 1 \rightarrow \|SW_p\|_\infty \leq 1$$

What can be captured?:

- minimal close loop bandwidth
- $\|S\|_\infty$
- Steady state tracking error

What cannot be captured?

- maximal bandwidth
- limitation on input signal

Mixed Sensitivity

Provide similar bounds for $T(s)$ → for maximal bandwidth
for $K(s)S(s)$ → for input signal magnitude

Notice that: $K(s)S(s)$ is the transfer function from r to u

We stack these requirements together:

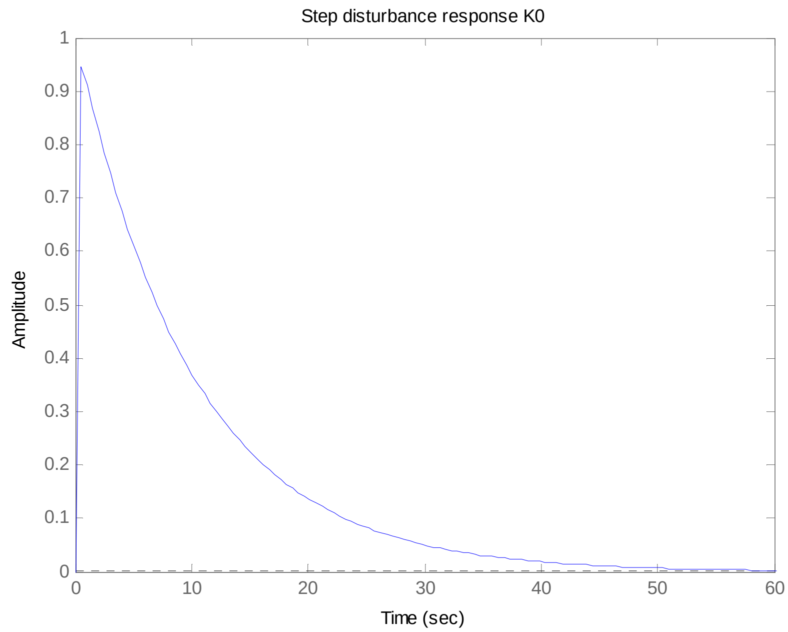
$$N(s) = \begin{bmatrix} W_P S \\ W_T T \\ W_u K S \end{bmatrix}$$

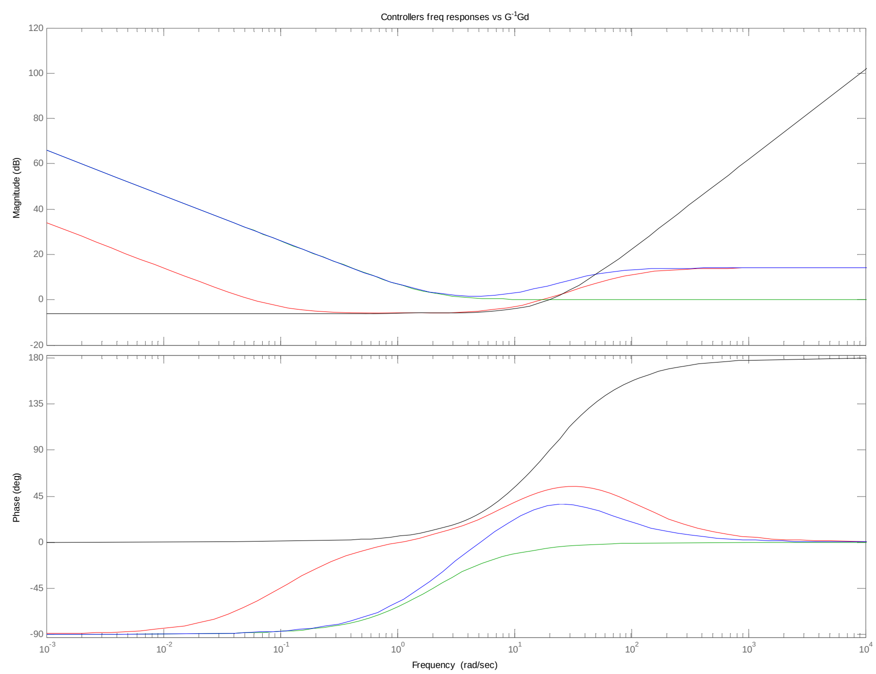
$$\|N\|_\infty = \sup_\omega \left(|W_P(j\omega)S(j\omega)|^2 + |W_T(j\omega)T(j\omega)|^2 + |W_u(j\omega)K(j\omega)S(j\omega)|^2 \right)^{1/2}$$

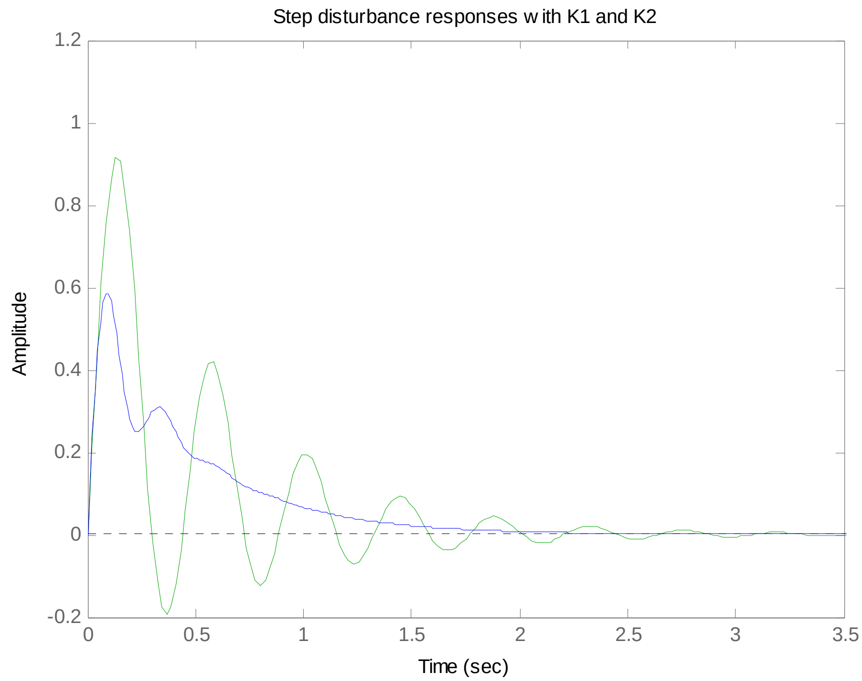
Thus if $\|N\|_\infty \leq 1$, all the bounds are satisfied.

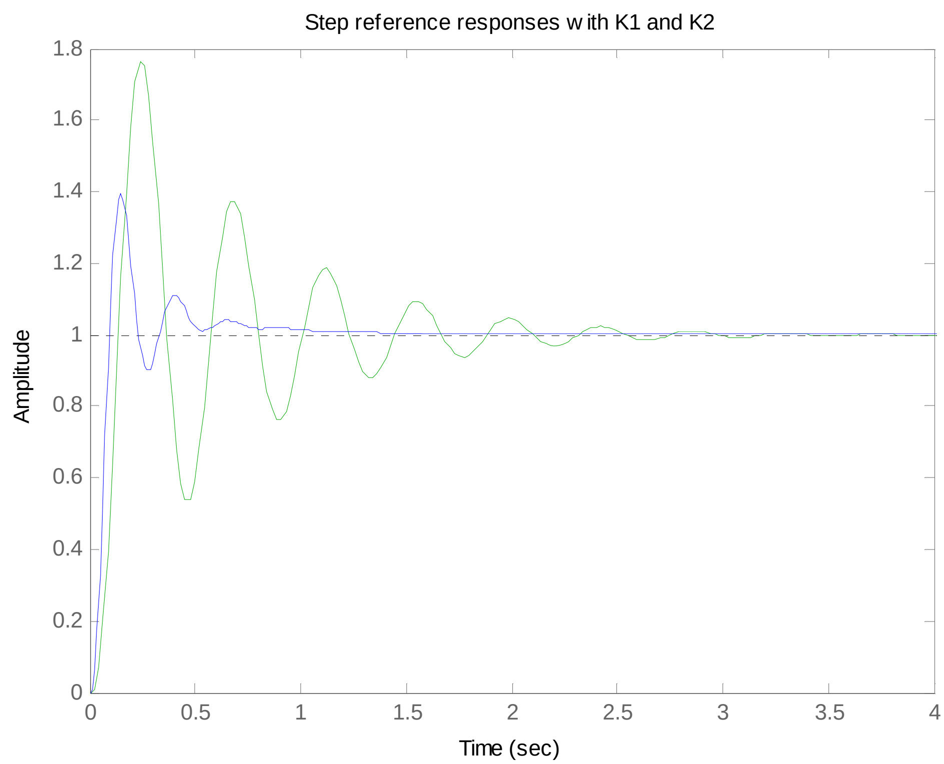
In MATLAB H_∞ controller synthesis is done with "mixsyn"

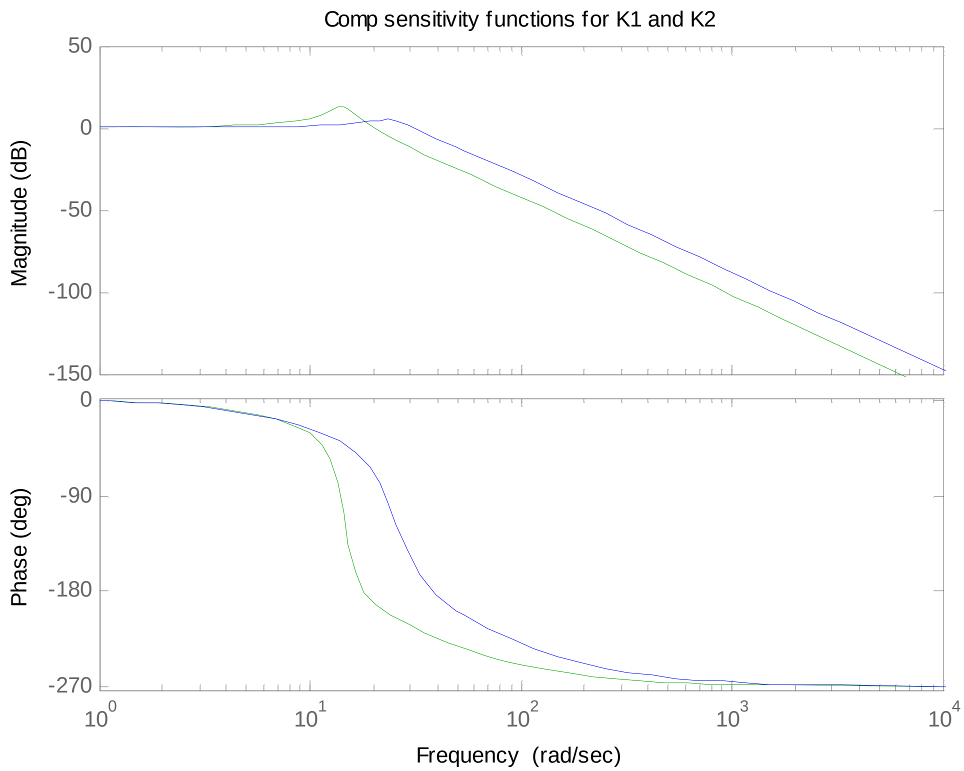
Read example 2.17

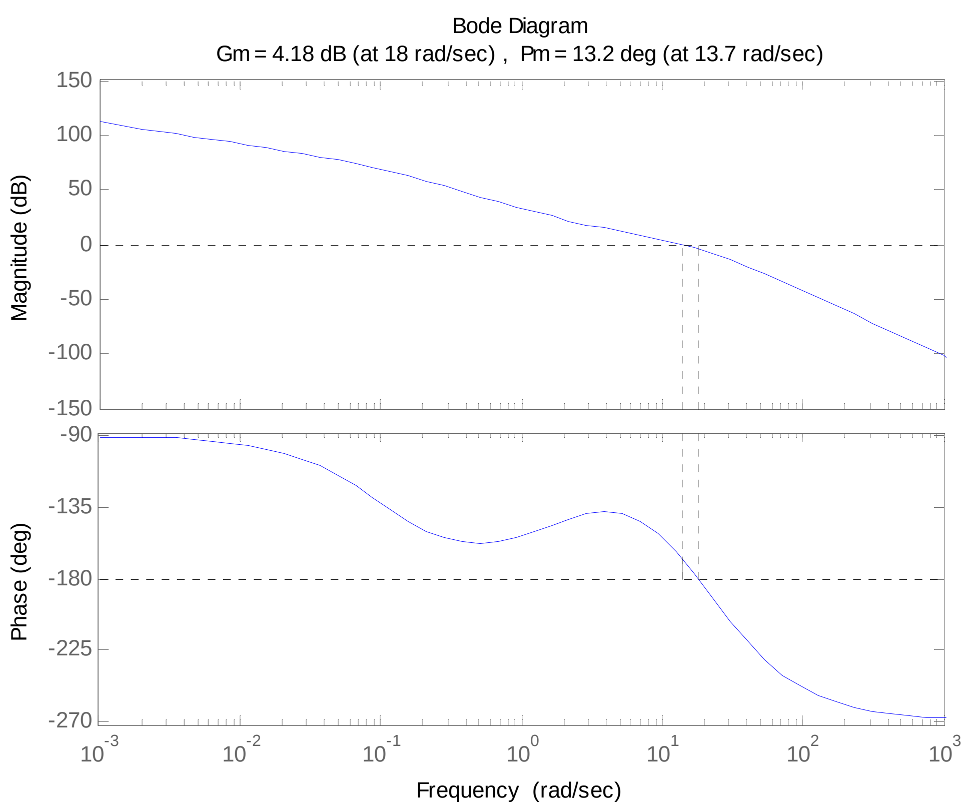

Step disturbance response K0
Amplitude
Time (sec)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60

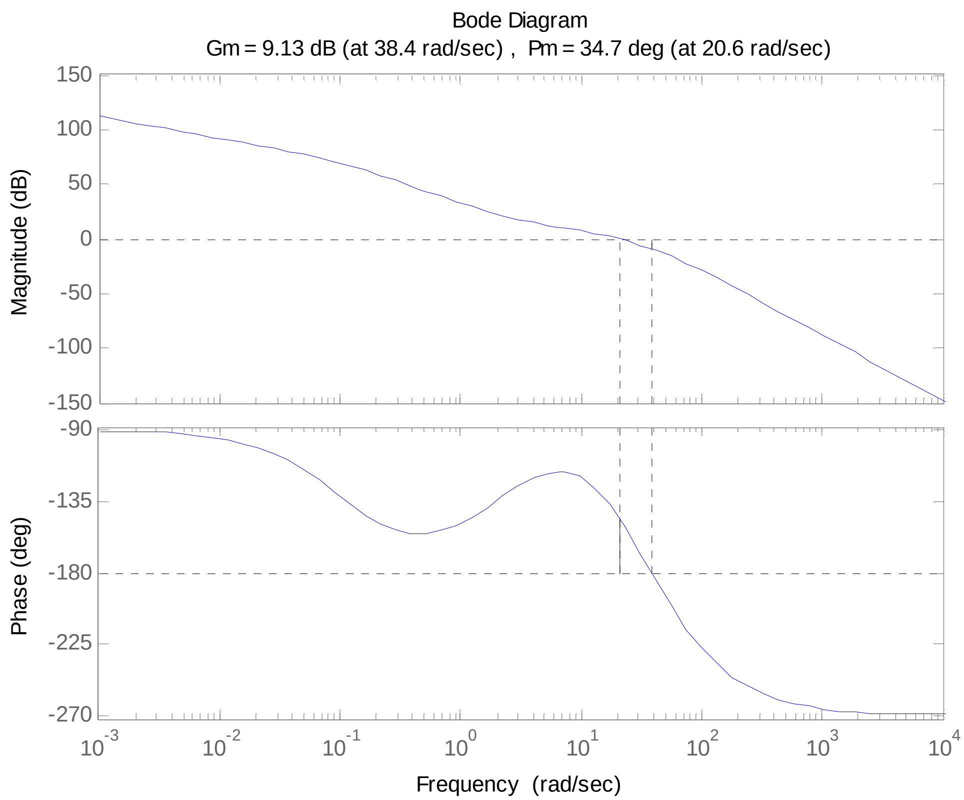

Controllers freq responses vs G-1Gd
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

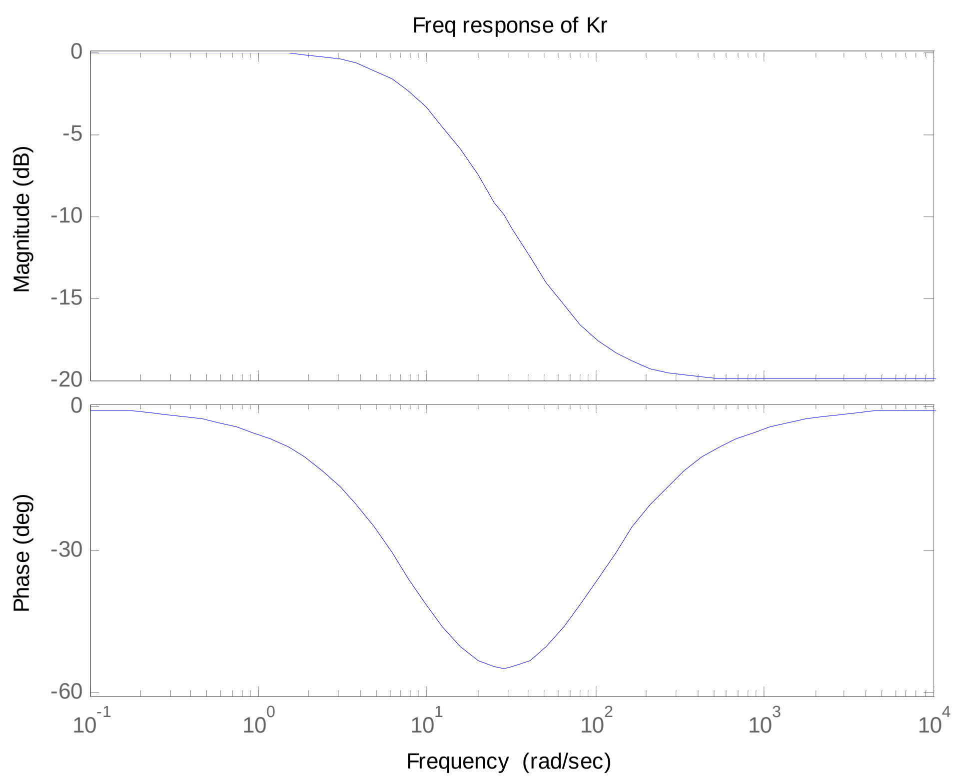

Step disturbance responses w ith K1 and K2
Amplitude
Time (sec)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.5
1
1.5
2
2.5
3
3.5

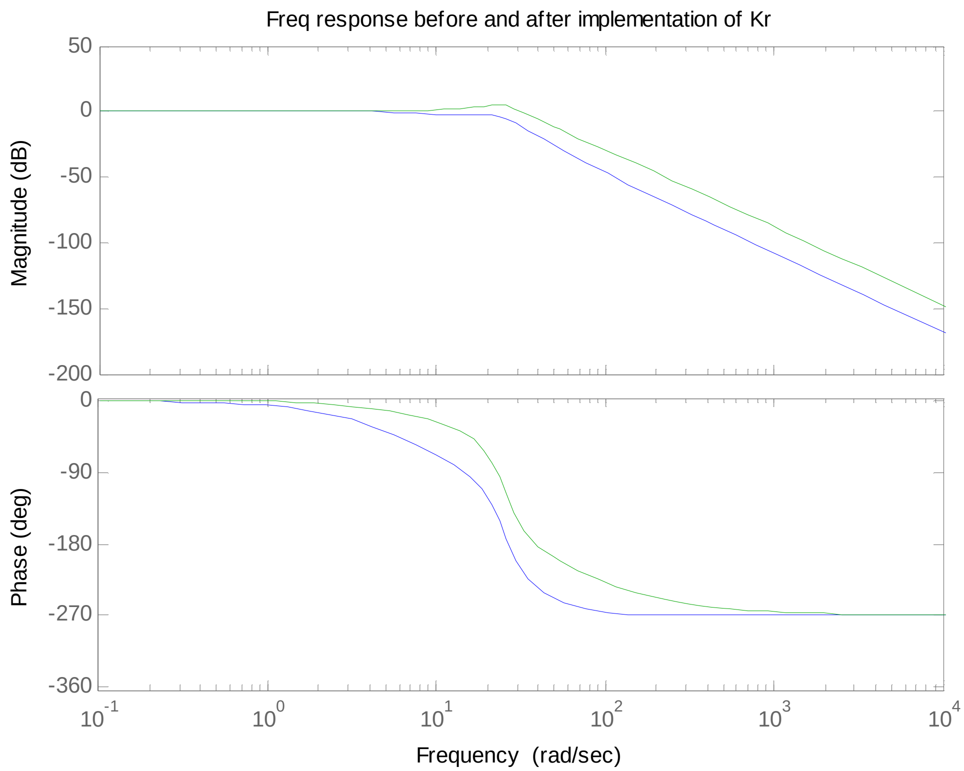

Step reference responses w ith K1 and K2
Amplitude
Time (sec)
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4

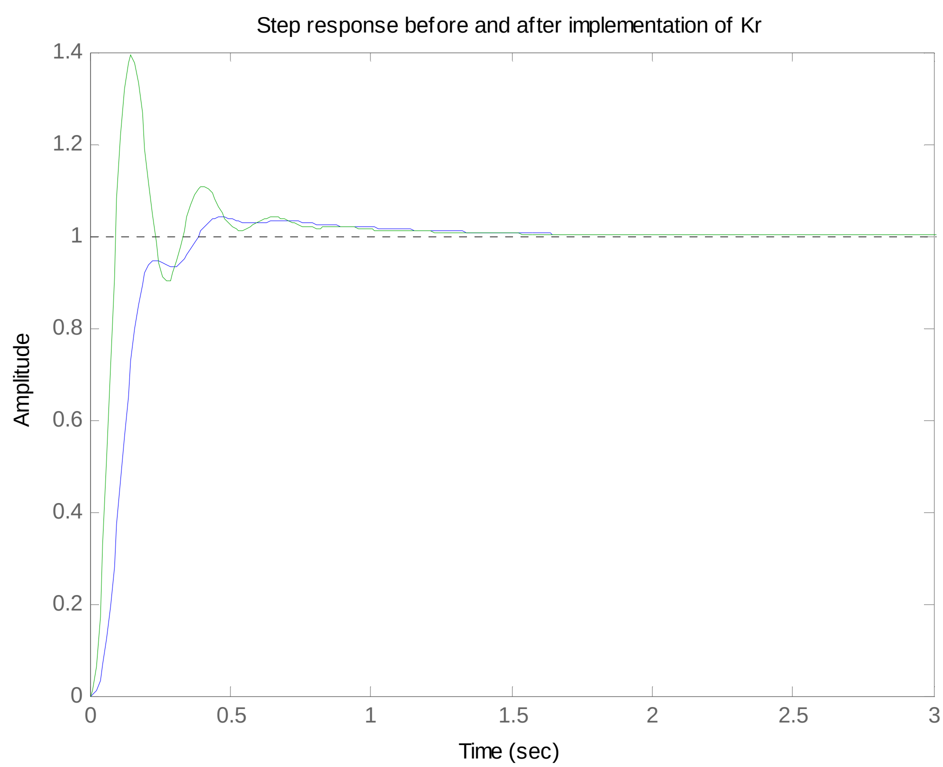

Comp sensitivity functions for K1 and K2
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
50
0
-50
-100
-150
-90
-180
-270
10^0
10^1
10^2
10^3
10^4


Bode Diagram
Gm = 4.18 dB (at 18 rad/sec) , Pm = 13.2 deg (at 13.7 rad/sec)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
150
100
50
0
-50
-100
-150
-90
-135
-180
-225
-270
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
10^2
10^3


Bode Diagram
Gm = 9.13 dB (at 38.4 rad/sec) , Pm = 34.7 deg (at 20.6 rad/sec)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)


Freq response of Kr
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)


Freq response before and after implementation of Kr
50
0
-50
-100
-150
-200
Magnitude (dB)
0
-90
-180
-270
-360
Phase (deg)
10-1
100
101
102
103
104
Frequency (rad/sec)


Step response before and after implementation of Kr
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Amplitude
0
0.5
1
1.5
2
2.5
3
Time (sec)