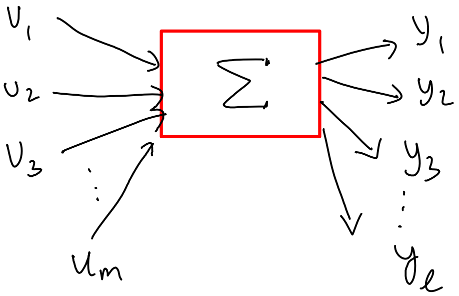
Intro to MIMO Systems



In Freq. Domain / Transfer Function:

$$Y_1 = G_{11}u_1 + G_{12}u_2 + \cdots + G_{1m}u_m$$

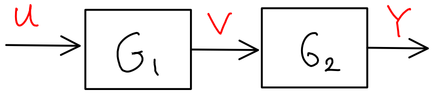
$$\vdots$$

$$\underbrace{Y_\ell = G_{\ell 1}u_1 + G_{\ell 2}u_2 + \cdots + G_{\ell m}u_m}_{Y(s) = G(s)\,U(s)}$$

Since $G(s)$ is a matrix, in general multiplications don't commute:

i.e $G_1 G_2 \neq G_2 G_1$

Examples / rules:

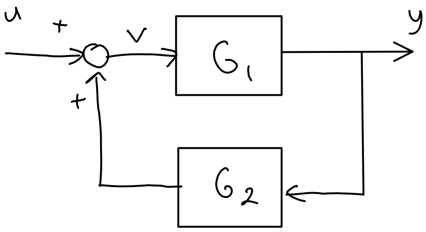


$$V = G_1 u$$

$$Y = G_2 V = G_2 G_1 u$$

(serial connections are read backward)

Positive Feedback:



$$Y = G_1 V = G_1(U + G_2 Y)$$

$$= G_1 U + G_1 G_2 Y$$

$$(I - G_1G_2)Y = G_1 U$$

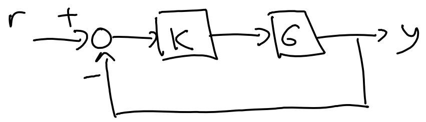
$$Y = (I - G_1G_2)^{-1} G_1 U$$

Push-through Identity: $(I - G_1G_2)^{-1}G_1 = G_1(I - G_2G_1)^{-1}$

Proof: $G_1(I - G_2G_1) = (I - G_1G_2)G_1$

$$(I+G_1G_2)^{-1}G_1(\cancel{I-G_2}G_1)(\cancel{I+}G_2G_1) = (I\cancel{+G_1G_2})^{-1}(\cancel{I-}G_1G_2)G_1(I+G_2G_1)$$

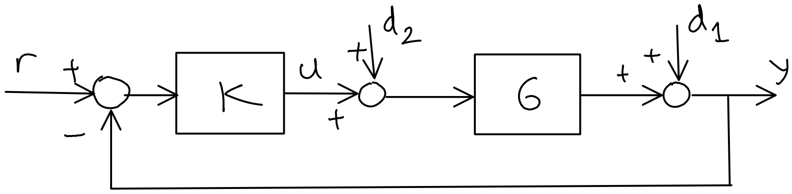
Application of Positive Feedback Rule: $G_1 = GK$



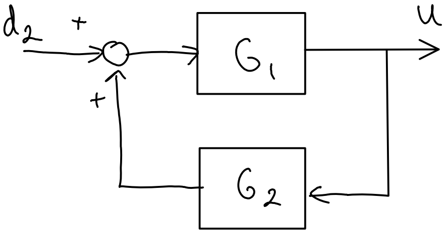
$$G_2 = -I$$

$$Y = (I + GK)^{-1} GK R$$

$$= GK(I + GK)^{-1} R$$



We can apply the feedback rule to compute various transfer functions



Ex: from d_2 to u

Clearly: $G_2 = I$

$G_1 = -KG$

Thus the transfer function is $-(I+KG)^{-1}KG$

Define:

$L = GK$

$S = (I+L)^{-1}$

$T = (I+L)^{-1}L$

$= L(I+L)^{-1}$

$L_I = KG$

$S_I = (I+L_I)^{-1}$

$T_I = (I+L_I)^{-1}L_I = L_I(I+L_I)^{-1}$

Read useful identities (p.70)

Multivariable Freq. Response

$d = d_1, \dots, d_m$

$y = y_1, \dots, y_\ell$

Suppose that $d_1, \dots, d_m$ are all sinusoidal signals with frequency ω, $d_1 = d_{10}\sin(\omega t + \alpha_1)$; $d_2 = d_{20}\sin(\omega t + \alpha_2), \dots$ $d_m = d_{m0}\sin(\omega t + \alpha_m)$

then $y_{1\dots\ell}$ are also sinusoidal signals with frequency ω:

$$y_1 = y_{10}\sin(\omega t + \beta_1)$$

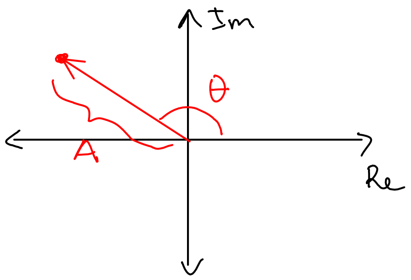
$$y_2 = y_{20}\sin(\omega t + \beta_2)$$

$$\vdots$$

$$y_\ell = y_{\ell 0}\sin(\omega t + \beta_\ell)$$

Use phasor notation:

$$A\sin(\omega t+\theta) \Longrightarrow A\cdot e^{j\theta} = A\cos\theta + jA\sin\theta$$



$$y_{10}\, e^{jB_1} = \sum_{K=1}^{m} G_{1K}(j\omega)\cdot d_{K0} e^{j\alpha_K}$$

$$\vdots$$

$$y_{l0}\, e^{jB_l} = \sum_{K=1}^{m} G_{lK}(j\omega)\, d_{K0} e^{j\alpha_K}$$

$$\underset{\substack{\uparrow \\ \text{phasor} \\ \text{notation}}}{y(\omega)} = G(j\omega)\underset{\substack{\nwarrow \\ \text{phasor} \\ \text{notation}}}{d(\omega)}$$

Example: $G(s) = \begin{bmatrix} \frac{1}{s} & s \\ 1 & \frac{1}{s+1} \end{bmatrix}$; $d_1(t) = \sin t$

$d_2(t) = \cos t = \sin(\omega t + \frac{\pi}{2})$

$\omega = 1.$

Phasor notation: $d_1(\omega) = 1$;

$d_2(\omega) = j$

$$G(j\omega) = \begin{bmatrix} \frac{1}{j} & j \\ 1 & \frac{1}{j+1} \end{bmatrix} = \begin{bmatrix} -j & j \\ 1 & \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right. \end{bmatrix} = \begin{bmatrix} -j & j \\ 1 & \frac{1}{2} - j\frac{1}{2} \end{bmatrix}$$

Thus: $\begin{bmatrix} y_1(\omega) \\ y_2(\omega) \end{bmatrix} = \begin{bmatrix} -j & j \\ 1 & \frac{1}{2} - \frac{1}{2}j \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix} = \begin{bmatrix} -1-j \\ \frac{3}{2} + \frac{1}{2}j \end{bmatrix}$

$|y_1(\omega)| = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$; $\angle y_1(\omega) = 225^\circ = \frac{5}{4}\pi$

$|y_2(\omega)| = \sqrt{\frac{9}{4} + \frac{1}{4}} = \frac{1}{2}\sqrt{10}$; $\angle y_2(\omega) = \arctan\frac{1}{3} = 0.32$ rad

$y_1(t) = \sqrt{2}\sin(\omega t + 5/4)$; $y_2(t) = \frac{1}{2}\sqrt{10}\sin(\omega t + 0.32)$

System Gain

We consider the vector 2-norm of the signals:

$$\left.\begin{array}{l} \|d(\omega)\| = \sqrt{d_{10}^2 + d_{20}^2 + \cdots + d_{m0}^2} \\ \\ \|y(\omega)\| : \sqrt{y_{10}^2 + y_{20}^2 + \cdots + y_{\ell 0}^2} \end{array}\right\} \text{consider } d(\omega) \text{ and } y(\omega) \text{ as vectors of complex numbers, then:}$$

$$\|d(\omega)\| = \left(d^H(\omega)\, d(\omega)\right)^{1/2} \;;\; \|y(\omega)\| = \left(y^H(\omega)\, y(\omega)\right)^{1/2}$$

Recall that for $A \in \mathbb{C}^{p \times q}$, $A^H_{mn} = \bar{A}_{nm}$ ← complex conjugate

$$\text{System gain} = \frac{\|y(\omega)\|}{\|d(\omega)\|} = \frac{\|G(j\omega)\, d(\omega)\|}{\|d(\omega)\|}.$$

Notice that scaling $d(\omega)$ by a (complex) number does not change the gain, thus it only depends on the direction of $d(\omega)$

For SISO system, we recover that the system gain is $|G(j\omega)|$

Example: consider the previous example: $\|d(\omega)\| = \sqrt{2}$

$$\|y(\omega)\| = \sqrt{2 + \tfrac{10}{4}} = \tfrac{3}{2}\sqrt{2}$$

The gain is $\frac{3}{2}$, but if, for example $d_2(t) = \sin t$

$$d_2(\omega) = 1$$

$$\begin{bmatrix} y_1(\omega) \\ y_2(\omega) \end{bmatrix} = \begin{bmatrix} -j & j \\ 1 & \frac{1}{2} - \frac{1}{2}j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} - \frac{1}{2}j \end{bmatrix} \rightarrow \|y(\omega)\| = \sqrt{\tfrac{10}{4}}$$

The gain is only: $\sqrt{\frac{10}{4}} \cdot \sqrt{\frac{1}{2}} = \sqrt{\frac{10}{8}} \approx 1.12$

We are interested in the maximum and minimum gain for a given frequency ω.

$$\bar{\sigma}(G(j\omega)) = \max \frac{\|G(j\omega)\, d(\omega)\|}{\|d(\omega)\|} = \max_{\|d\|=1} \|G(j\omega)\, d(\omega)\|$$

$$\underline{\sigma}(G(j\omega)) = \min_{\|d\|=1} \|G(j\omega)\, d(\omega)\|$$

Singular Value Decomposition

Decompose $G(j\omega) \in \mathbb{C}^{l \times m}$ into $U \Sigma V^H$, where

$U \in \mathbb{C}^{l \times l}$ is a unitary matrix $U^H U = I = U U^H$
$V \in \mathbb{C}^{m \times m}$ is a unitary matrix $V^H V = I = V V^H$
$\Sigma \in \mathbb{C}^{l \times m}$ matrix with nonnegative main diagonal, sorted in descending order

Idea: The columns of U form an orthogonal basis in $\mathbb{C}^{l}$
— " — V — " — $\mathbb{C}^{m}$
The main diagonal of Σ provides the gain for each basis vector

$\bar{\sigma}(G(j\omega)$ is the largest entry of the main diagonal of Σ
$\underline{\sigma}(G(j\omega))$ is the smallest entry — " —
How to compute SVD? (HW problem)

Example: $G(j\omega) = \begin{bmatrix} -j & j \\ 1 & \frac{1}{2} - \frac{1}{2}j \end{bmatrix}$ (use 'svd' in MATLAB)

$$U = \begin{bmatrix} 0.26 + 0.77j & -0.58 \\ -0.52 - 0.26j & -0.58 + 0.58j \end{bmatrix}; \quad \Sigma = \begin{bmatrix} 1.58 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.82 & -0.58 \\ 0.41 - 0.41j & -0.58 + 0.58j \end{bmatrix}$$

Thus, the gain is maximized if $d(\omega)$ is aligned with $\begin{bmatrix} -2 \\ 1-j \end{bmatrix}$, and the gain is 1.58. Similarly with the minimum gain.