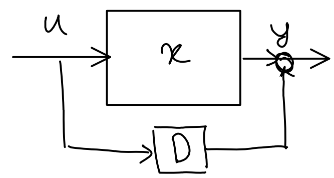
State Space Representation (Ch. 4)

SISO & MIMO systems can be represented in state space form:

$$\dot{x} = Ax + Bu \quad , \; x \in \mathbb{R}^n , \; u \in \mathbb{R}^m$$
$$y = Cx + Du \quad , \; y \in \mathbb{R}^\ell$$

conceptually:



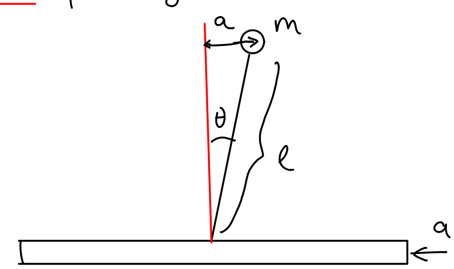
Transfer Function:

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$X(s) = (sI - A)^{-1} B\, U(s)$$
$$Y(s) = \left[C(sI - A)^{-1} B + D \right] U(s)$$

The polynomial $\chi(s) = \det(sI - A)$ is called the characteristic polynomial of the system. The roots of $\chi(s)$ are the poles of the system.

Example: Inverted pendulum



$$\cancel{m}\ell^{\cancel{2}}\,\ddot{\theta} = \cancel{m}a \cdot \cancel{\ell}\cos\theta + \cancel{m}g\cancel{\ell}\sin\theta$$

$$\ell\ddot{\theta} = a\cos\theta + g\sin\theta$$

For θ small: $\cos\theta \approx 1$; $\sin\theta \approx x$ ← Taylor expansion / linearization

$$\boxed{\ell\ddot{\theta} = a + g\theta} \rightarrow \ell s^2 \theta(s) = A(s) + g\theta(s)$$

$$\theta(s) = \frac{A(s)}{\ell s^2 - g}$$

State space representation:

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

$$
\left.
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{g}{l} x_1 + \frac{1}{l} u \\
y &= x_1
\end{aligned}
\right\}
\quad
A = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix}; \; B = \begin{pmatrix} 0 \\ 1/l \end{pmatrix}; \; C = [1 \quad 0]; \; D = 0
$$

$$
(sI - A) = \begin{bmatrix} s & -1 \\ -g/l & s \end{bmatrix} \rightarrow \det(sI - A) = s^2 - g/l \rightarrow \text{poles} = \pm\sqrt{g/l}
$$

However, given a transfer function, there is not a unique state space representation. Why?

Similarity Transformation

If $z = Tx$, $T \in \mathbb{R}^{n \times n}$ invertible matrix, then

$$
\dot{z} = T\dot{x} = TAx + TBu = TAT^{-1}z + TBu
$$

$$
y = Cx + Du = CT^{-1}z + Du
$$

Thus:

$$
\begin{aligned}
Z(s) &= (sI - TAT^{-1})^{-1} TBu \\
Y(s) &= CT^{-1}(sI - TAT^{-1})^{-1} TB\, u(s) + Du(s) \\
&= C\left(T^{-1}(sI - TAT^{-1})T\right)^{-1} Bu(s) + Du(s) \\
&= C(sI - A)^{-1} Bu(s) + Du(s)
\end{aligned}
$$

Nonminimum Representation

Take the pendulum example. Add an extra state, e.g.:

$$
\left.
\begin{aligned}
\dot{x}_3 &= x_3 \\
y &= x_1 + x_3
\end{aligned}
\right\}
\quad
A = \begin{bmatrix} 0 & 1 & 0 \\ g/l & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 1/l \\ 0 \end{bmatrix}
$$

We can see that the transfer function from u to y does not change. However, the characteristic equation now becomes:

$$
\det(sI - A) = (s-1)(s^2 - g/l)
$$

or alternatively:

$$
\begin{aligned}
\dot{x}_3 &= \cdot x_3 + u \\
y &= x_1
\end{aligned}
$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ g/\ell & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 1/\ell \\ 1 \end{bmatrix} ; \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Same char. polynomial as previous example

State controllability

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

Definition: $p \in \mathbb{R}^n$ is controllable if there is an input signal $u(\cdot)$ that brings the state from 0 to p in finite time, i.e.

$$p = \int_0^T e^{A(T-\tau)} B u(\tau)\, d\tau$$

Remark: the definition does not require the state to stay at p.

Property: • If p is controllable, then λp is controllable for any $\lambda \in \mathbb{R}$

• If p & q are controllable, then $p+q$ is controllable

Thus: the set of controllable states is a linear space $\mathcal{C}$

If the set $\mathcal{C} = \mathbb{R}^n$, then the system is controllable

$$\mathcal{C} = \operatorname{im} \left[B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B \right]$$

Example: $A = \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix} ; B = \begin{bmatrix} 0 \\ 1/\ell \end{bmatrix} \rightarrow [B \,\vdots\, AB] = \begin{bmatrix} 0 & 1/\ell \\ 1/\ell & 0 \end{bmatrix}$

rank $= 2$, $\operatorname{im}[B \,\vdots\, AB] = \mathbb{R}^2$, the system is controllable

$$A = \begin{bmatrix} 0 & 1 & 0 \\ g/\ell & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 1/\ell \\ 0 \end{bmatrix}$$

$[B \,\vdots\, AB \,\vdots\, A^2B] = \begin{bmatrix} 0 & 1/\ell & 1/\ell \\ 1/\ell & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ rank $= 2$ (uncontrollable)

$$\mathcal{C} = \operatorname{im}\begin{bmatrix} 0 & 1/\ell & 1/\ell \\ 1/\ell & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$, thus the third state is not controllable

How to compute the uncontrollable space? USE SVD

$\mathcal{C} = USV^T$, the uncontrollable space is spanned by the columns of U corresponding to singular value $= 0$.

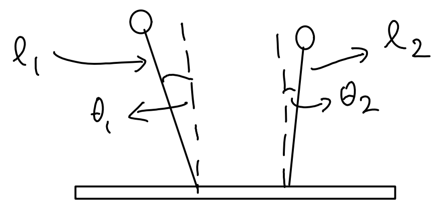
If $\dot{x} = Ax + Bu$ is not controllable, we can decompose the state space by similarity transformation: $z = Tx$, such that

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix} u$$

z_2 is the uncontrollable part of the state space. In this case we can "minimize" the dimension of the state space by removing the z_2 part, without changing the I/O behavior.

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \tilde{B} u$$

(counter intuitive example): Consider a double inverted pendulum system:



$x_1 = \theta_1$; $x_3 = \theta_2$

$x_2 = \dot{\theta}_1$; $x_4 = \dot{\theta}_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ g/\ell_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/\ell_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/\ell_1 \\ 0 \\ 1/\ell_2 \end{bmatrix} u$$

$$[B \vdots AB \vdots A^2B \vdots A^3B] = \begin{bmatrix} 0 & 1/l_1 & 0 & g/l_1^2 \\ 1/l_1 & 0 & g/l_1^2 & 0 \\ 0 & 1/l_2 & 0 & g/l_2^2 \\ 1/l_2 & 0 & g/l_2^2 & 0 \end{bmatrix}$$

Fullrank iff $\begin{bmatrix} 1/l_1 & g/l_1^2 \\ 1/l_2 & g/l_2^2 \end{bmatrix}$ is fullrank

$$\text{determinant} = \frac{g}{l_1 l_2^2} - \frac{g}{l_1^2 l_2} = \frac{g}{l_1 l_2}\left(\frac{1}{l_1} - \frac{1}{l_2}\right)$$

If $l_1 \neq l_2$ the system is controllable!

If $l_1 = l_2$, the system is constrained that $x_1 = x_3$

$x_2 = x_4$