Implications of controllability: Pole Placement

Given a plant: $\dot{x} = Ax + Bu$, open loop poles are the eigenvalues of A: $\det(sI - A) = 0$ $\leftarrow$ eigenvalues of A

Using linear feedback $u = -Kx$, we can try to move the poles in closed loop: $\dot{x} = (A - BK)x = \tilde{A}x$

Closed loop poles $= \det(sI - \tilde{A}) = 0$ $\leftarrow$ eigenvalues of $\tilde{A}$

Theorem: Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, (A, B) is controllable. For any polynomial of order n, $p(s)$, there is a linear feedback $K \in \mathbb{R}^{m\times n}$ such that $\det(sI - A + BK) = p(s)$

How to compute K? (MATLAB 'acker' or 'place')

Single input case: suppose that $m = 1$.

* System in controllable canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ a_0 & a_1 & \cdots & & a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

openloop poles: $\det(sI - A) = s^n - a_{n-1}s^{n-1} - \cdots - a_0 = 0$

Suppose that $K = [k_0 \; k_1 \; \cdots \; k_{n-1}]$

$$A - BK = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ a_0 - k_0 & a_1 - k_1 & \cdots & & a_{n-1} - k_{n-1} \end{bmatrix}$$

Close-loop poles: $\det(sI - A + BK) = s^n + (k_{n-1} - a_{n-1})s^{n-1} + \cdots + (k_0 - a_0) = 0$

By matching the coefficients of the polynomials, the linear feedback is determined

- General case :

Suppose that we want $\lambda \in \mathbb{C}$ to be an eigenvalue of $\tilde{A}$, there must be an eigenvector v such that

$$(\lambda I - \tilde{A})v = 0 \quad \text{, or}$$
$$(\lambda I - A + BK)v = 0 \quad \text{, or}$$

$$\boxed{\begin{bmatrix} \lambda I - A & \vdots & B \end{bmatrix} \begin{bmatrix} v \\ Kv \end{bmatrix} = 0}$$

Hautus result : controllability $\Leftrightarrow$ rank $\begin{bmatrix} \lambda I - A & \vdots & B \end{bmatrix} = n$, $\forall \lambda \in \mathbb{C}$

Algorithm :

- For each desired close loop poles , compute $w_i = \begin{bmatrix} v_i \\ Kv_i \end{bmatrix}$ s.t

$$\begin{bmatrix} \lambda_i I - A & \vdots & B \end{bmatrix} w_i = 0$$

- We obtain : $W = \begin{bmatrix} w_1 & \vdots & w_2 & \vdots & \dots & \vdots & w_n \end{bmatrix} = \begin{bmatrix} v_1 & \vdots & v_2 & \vdots & \cdots & \vdots & v_n \\ Kv_1 & \vdots & Kv_2 & \vdots & & \vdots & Kv_n \end{bmatrix}$

$$K = \begin{bmatrix} Kv_1 & \vdots & Kv_2 & \vdots & \dots & \vdots & Kv_n \end{bmatrix} \begin{bmatrix} v_1 & \vdots & v_2 & \vdots & \dots & \vdots & v_n \end{bmatrix}^{-1}$$

Example : $A = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; Desired closed loop poles = $-3, -4$

open loop poles : $\det \begin{pmatrix} s & -2 \\ 0 & s-3 \end{pmatrix} = s(s-3) = 0$, 0 and -3

Applying the method:

$$[-3 \cdot I - A \ \vdots \ B] = \begin{bmatrix} -3 & -2 & 0 \\ 0 & -6 & 1 \end{bmatrix} w_1 = 0$$

$$w_1 = [-\tfrac{2}{3} \ \ 1 \ \ 6]^T$$

$$[-4I - A \ \vdots \ B] = \begin{bmatrix} -4 & -2 & 0 \\ 0 & -7 & 1 \end{bmatrix} w_2 = 0$$

$$w_2 = [-\tfrac{1}{2} \ \ 1 \ \ 7]^T$$

Thus: $$K = [6 \ \ 7] \begin{bmatrix} -\frac{2}{3} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}^{-1} = [6 \ \ 10]$$

Multiple inputs $(m > 1)$

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 + \dots B_m u_m$$

$B_1, B_2, \dots, B_m$ are vectors.

Each input has its own controllable space:

$$\mathcal{C}_i = \text{im}\,[B_i \ \vdots \ AB_i \ \vdots \ A^2 B_i \cdots \vdots \ A^{n-1} B_i]$$

If for any i, $\dim \mathcal{C}_i = n$, then the system is already controllable with that input alone. We can set all other inputs to zero.

Thus we are focusing our attention on the situation where the individual controllable spaces contribute to $\mathbb{R}^n$.

For discussion sake, let's assume $m=2$, then generalize. Since $m=2$, we know that $\dim \mathcal{C}_1 < n$, and $\dim \mathcal{C}_2 < n$, but

$$\dim(\mathcal{C}_1 \oplus \mathcal{C}_2) = n \quad \text{or}$$

$$\text{rank}\,[B_1 \ \vdots \ AB_1 \ \vdots \cdots \vdots \ A^{n-1}B_1 \ \vdots \ B_2 \ \vdots \ AB_2 \ \vdots \cdots \vdots \ A^{n-1}B_2] = n$$

Decompose the state space into $\mathcal{C}_1$ and $\bar{\mathcal{C}}_1$ using similarity transformation:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} \tilde{B}_{21} \\ \tilde{B}_{22} \end{pmatrix} u_2$$

Notice that $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable

$(\tilde{A}_{22}, \tilde{B}_2)$ is controllable

Design the feedback: $u_1 = -K_1 z_1 \rightarrow u_1 = [-K_1 \quad 0] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$u_2 = -K_2 z_2 \rightarrow u_2 = [0 \quad -K_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

Close loop system:
$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} \tilde{B}_1 K_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} 0 & \tilde{B}_{21} K_2 \\ 0 & \tilde{B}_{22} K_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{A}_{11} - \tilde{B}_1 K_1 & \tilde{A}_{12} - \tilde{B}_{21} K_2 \\ 0 & \tilde{A}_{22} - \tilde{B}_{22} K_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Because of the block triangular structure, the close loop poles are given by: $\det(sI - \tilde{A}_{11} + \tilde{B}_1 K_1) \cdot \det(sI - \tilde{A}_{22} + \tilde{B}_{22} K_2) = 0$

Thus, assign some of the poles with u_1, and the rest with u_2.

Observability

Another way to obtain non-minimal representation is to add states do not matter to the output. Recall the inverted pendulum example:

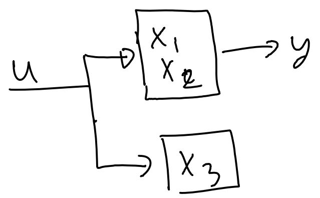
$A = \begin{pmatrix} 0 & 1 \\ g/\ell & 0 \end{pmatrix}$; $B = \begin{pmatrix} 0 \\ 1/\ell \end{pmatrix}$; $C = [1 \quad 0]$

Consider the following representation:

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ g/\ell & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad \hat{B} = \begin{bmatrix} 0 \\ 1/\ell \\ 1 \end{bmatrix} ; \quad \hat{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathcal{C} = \operatorname{im} \begin{bmatrix} 0 & 1/\ell & 0 \\ 1/\ell & 0 & g/\ell^2 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \text{rank is 3 if } g \neq \ell \text{, thus controllable}$$

However it is obvious that the third state does not matter to the output.



Definition: a state $p \in \mathbb{R}^n$ is observable if, assuming that $u(t) \equiv 0$, $x(0) = p$ implies $y(t) \neq 0$ for some $t \geq 0$.

It's quite easy to show that the set of all observable states is also a linear subspace

$$\mathcal{O} = \operatorname{im} \left[C^T \,\vdots\, AC^T \,\vdots\, \cdots \,\vdots\, A^{n-1} C^T \right]$$

With similarity transformation, we can decompose $\mathbb{R}^n$ into $\mathcal{O}$ and $\bar{\mathcal{O}}$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$