Optimal Precooling of Thermostatic Loads under Time-varying Electricity Prices

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Abstract— **In this work, we consider the problem of optimal precooling of thermostatic loads by a load serving entity (LSE) under time varying electricity prices. Specifically, we propose a low complexity dynamic programming based algorithm for controlling the switching of HVAC units such that the cost of cooling is minimized while ensuring that the resultant temperatures do not violate the comfort constraints of consumers. Specifically, our algorithm makes use of two facts: (i) the price of electricity vary discretely in time and (ii) the cost-optimal state trajectory under fixed electricity price and ambient temperature, with given initial and final states, can be characterized explicitly in closed-form. We validate our algorithm with a set of numerical simulation studies using real market prices of electricity from the New York electricity market.**

I. Introduction

Heating, Ventilation and Cooling (HVAC) systems amount for a significant fraction of the overall energy consumption in our day-to-day life. In the United States, for example, about 40% of the total energy consumption of buildings is due to HVAC systems. During very hot days in summer months, cooling requirements can increase substantially, causing a considerable rise in the peak power consumption in the grid. Increased demand for electricity may require uneconomical ramp up of expensive generators resulting in increased real-time electricity prices, which can rise as high as 100 times the average value. Under such cases, load serving entities (LSE) who buy the energy in the wholesale power market and dispatch it to residential (or commercial) consumers may have to procure the energy at very high prices. To avoid such uneconomical operation, Demand Response (DR) utilizing building HVAC systems has been looked upon as a possible solution.

In practice, several utilities have already initiated DR programs (such as the SmartAC program [1] by Pacific Gas & Electric (PG&E)) that allows the utility to centrally control building thermostats under emergent grid conditions during summer months. Most of these programs curtail the load during specific event days, during which grid may be stressed due to higher-than-normal demand for electricity. Some programs offer Time-of-Use (ToU) pricing to offset the deferrable demand of consumers to times that have relatively lesser demand and hence lower prices. In the research community, there has been considerable work in recent years on designing sophisticated DR algorithms for scheduling HVAC loads to attain energy cost savings and/or energy efficiency.

Much of the research on this topic, as well as the DR programs deployed in practice, are *reactive* in nature. In other words, these solutions involve deferring HVAC load when grid overload conditions have been detected (possibly indicated by real-time electricity prices). Furthermore, these mechanisms usually trade off user comfort for cost savings. Since HVAC units can be shut off at times of high prices, these solutions can result in temperatures that are beyond the comfort range of the occupants. Even though such comfort level violations are temporary, unsuccessful DR projects such as the one by PECO [2] have stressed on the importance of considering occupant comfort in designing DR algorithms.

In this work, we consider the problem of designing DR algorithms for air-conditioning (AC) loads that are proactive in nature and do not violate the occupant-defined thermal comfort range at any time. More specifically, our approach utilizes electricity price forecasts to *precool* buildings (over summer months) such that the total cost of cooling is minimized. Our model and solution approach assumes that a typical user (occupant) will be insensitive to temperatures going down to a certain level below its desired temperature set-point ("precooling depth"), and exploits this fact to save cost through precooling while ensuring that the temperature never exceeds the occupant-defined set-point.

The specific contributions of this work are as follows. We pose the optimal precooling problem (assuming forecast prices) as a finite-horizon optimal control problem (Section III). Since the control variable is a continuous function of time, a direct solution of this optimal control problem is computationally complex. Despite this, we show (in Section IV) how a low-complexity solution can be obtained by exploiting two facts: (i) electricity prices (day-ahead or real-time) vary discretely in time, (ii) the cost-optimal state trajectory under fixed electricity price and ambient temperature, with given initial and final states, can be characterized explicitly in closed-form. The resulting algorithm is linear in the number of price variations over the time horizon under consideration, and quadratic in the number of temperature discretization levels of the occupant's thermal comfort range. Finally, we use this optimal solution to compute the potential for cost savings through precooling of buildings, using actual day-ahead and real-time electricity price data (Section V).

II. Prior Work

Demand response of HVAC load in buildings has been addressed in recent literature through various approaches,

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such as economic model predictive control [3], [4], Newton's method [5], dynamic pricing [6], [7], and game theoretic formulations [8]. In a related body of work, the potential for utilizing HVAC loads for providing regulation services to the smart grid has been evaluated in several recent papers, such as [9], [10].

Our work is most closely related to the recent literature that provides solutions to, or evaluates the potential of, precooling in buildings for providing cost-savings. Specifically, Turner *et. al.* in their work [11] did an experimental study to evaluate the potential for DR by mechanical precooling and shifting electrical loads away from peak demand periods. Cole et. al. in [12] have proposed decentralized architectures for residential air conditioning control to bring about effective grid load management. They have shown that for a utility based in Texas, under peak summer conditions, through effective control of HVAC loads in a community scale, upto 7% energy cost savings can be obtained. In [15], the authors present a data driven precooling strategy for energy cost minimization under time of use retail energy exposure in the Australian context. The authors in [13] also consider the problem of coordinating the HVAC energy usage with a view of optimizing the aggregator's cost of procuring the same energy from the wholesale market. They also propose a controller which enables the distributed HVAC devices to track the reference optimal signal for energy usage which is centrally computed by the aggregator. Finally, while our work is closely related to a recent work [14], there are significant differences. The authors in [14] consider an additional energy constraint which makes their problem formulation and optimal solution different from ours. Further, whereas [14] focuses on obtaining structural properties of the optimal control solution, our primary goal is to obtain computationally efficient solutions to the problem through the use of discrete time dynamic programming.

III. System Model and Problem Formulation

Consider a look-ahead time window $[0, W]$ for which electricity price forecast (assumed to be accurate) is available. Let the price of electricity at time t be denoted as $p(t)$. Consider an aggregator (also known as load serving entity (LSE)), and let there be N thermostatically controlled loads (TCL) under its purview. Each of these loads has a desired temperature set-point. We consider summer operation where the ambient temperature is typically higher than this desired temperature. We therefore consider AC operation alone (i.e., heating is not needed/considered), which must be operated such that the building indoor temperature never exceeds the desired set-point. In other words, the desired set point also represents the upper bound on the building occupant's comfort range, which must not be violated at any time. Under time-invariant prices, therefore, the cost-optimal AC operation strategy would be to keep the building at this desired set-point at all times, except when the ambient temperature falls below the set-point. We are interested to find out whether precooling the building (TCL) to temperatures below this desired set-point can save cost, when the electricity prices are time-varying. We assume that the building occupant has a limit on how low the temperature can be at any time, and this lower bound on the building indoor temperature cannot be violated at any time.

We assume that electricity prices are unaffected by the scheduling of the AC load - if the LSE is a relatively small participant in the wholesale electricity market, it is reasonable to assume that it will be a "price-taker". Under this assumption, and in absence of any other constraints (such as transformer or feeder capacity constraints) that may apply to the aggregate of the TCLs, the cost-optimal operation of the ACs for the N TCLs can be determined independently of each other. For the rest of the paper, therefore, we focus on the decoupled optimal precooling problem for a single TCL, denoted by z.

Let T_z^{des} and T_z^{lb} respectively denote the desired and and lower bound temperature for the TCL under consideration. Thus, $\Delta = T_z^{des} - T_z^{lb}$ denotes the precooling depth. Let the indoor temperature and external ambient temperature at an instant t be denoted as $T_z(t)$ and $T_\infty(t)$, respectively. We assume that the AC in question has a cooling power rating of $\bar{u}$, an instantaneous cooling power consumption of $u(t)$ and a coefficient of performance η[1]. We assume that the power output of the AC can vary continuously between 0 and $\bar{u}$, where intermediate values $0 < u < \bar{u}$ may be attained through duty cycling the compressor between ON (power output = $\bar{u}$) and OFF (power output = 0) states with very low amplitude and high frequency. Many AC units operate only in a binary (ON or OFF) modes, so such high-frequency duty cycling may be needed to maintain the building at a particular temperature. Let u_{avg}^{des}, where $0 < u_{\text{avg}}^{des} < \bar{u}$, be the average power output required to maintain the building at the desired temperature T_z^{des}. Note that u_{avg}^{des} is a function of the the desired temperature T_z^{des} as well as the ambient temperature T_∞. We assume a typical resistive-capacitative (R-C) model to represent the thermal dynamics of the building (cooling space associated with the TCL), and equivalent building thermal parameters for this TCL are denoted as R and C. We also assume that these thermal parameters are known (or well estimated) *apriori* by the LSE.

In such a setting, the cost minimizing precooling problem for LSE z can be defined as follows,

$$\min_{u(t)} \quad \mathcal{J}(u) = \frac{1}{\eta} \int_{t \in [0,W]} p(t)u(t)dt, \tag{1}$$

$$s.t. \quad C\dot{T}_z(t) = \frac{1}{R}\Big(T_\infty(t) - T_z(t)\Big) - u(t) \tag{2}$$

$$T_z(0) = T_z^0, \tag{3}$$

$$T_z^{lb} \le T_z \le T_z^{des}, \tag{4}$$

$$u(t) \in \mathcal{U}_t, \tag{5}$$

where $\mathcal{U}_t$ represents the set of all feasible $u(t)$; in our case $\mathcal{U}_t = [0, \bar{u}]$. Note that (2) represents the indoor temperature evolution dynamics, and (4) represents the comfort

[1] u kW of cooling power drawn from the HVAC would result into an effective electrical power of u/η kW to be drawn from the grid.

constraints for the consumer under consideration. To ensure that the AC has enough power to cool over the entire range of indoor temperatures $[T_z^{lb}, T_z^{des}]$, we assume $\bar{u} > \frac{T_\infty(t) - T_z^{lb}}{R}$ for all $t \in [0, W]$.

IV. Efficient Dynamic Programming based Solution

The question posed above is a challenging minimum-cost continuous-time optimal control problem which is difficult to solve directly in a computationally feasible manner. The reason for this is that the optimal control variable $\{u(t)\}$ (i) is a continuous function of time, and (ii) takes values over a continuous set $[0, \bar{u}]$. In this section, we show how these issues can be dealt with, towards developing an efficient (low-complexity) solution to the optimal control problem using a combination of discrete-time dynamic programming and explicit-form expressions for state transition trajectories (costs).

Towards addressing (i), we note that electricity prices change once every hour (for day-ahead prices) or once every five minutes or so (for real-time prices). Further, ambient temperature forecast data is usually available only at coarse time-scales (typically once every hour) and shows gradual variation in-between those data points. This naturally motivates us to consider the problem in the shortest time-scale at which the electricity price or the temperature forecast data varies. For day-ahead prices, an hourly time-scale may suffice; for real-time prices, we need to consider a time granularity of five minutes or so. Based on this time granularity, the look-ahead window $[0, W]$ is discretized into K distinct time slots $\{0, 1, 2, \ldots, k, \ldots, K-1\}$, each of duration $\tau = \frac{W}{K}$. Then, as we show below in Section IV-A, the continuous-time optimal control problem in (1)-(5) can be written as a dynamic programming problem in discrete time defined by these K time slots, where the electricity prices and ambient temperatures are assumed to remain constant over a time slot. Note that this does *not* imply that the control variable $u(t)$ varies only across time slots; in Section IV-B, we argue that the optimal control variable $u^*(t)$ can vary within a time slot, and show how it can be determined in explicit form.

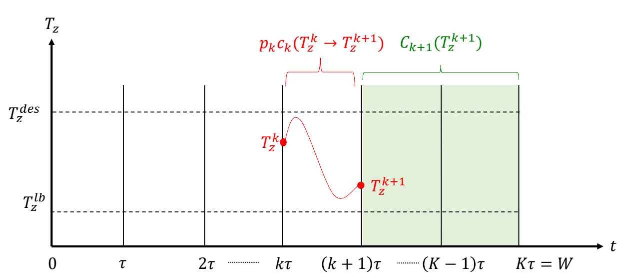

Fig. 1. Discrete-time dynamic programming approach to solve the optimization problem given in (1)-(5), that exploits the fact that the electricity prices only vary discretely over time.

A. Backward Induction

We now describe the recursion relation for the discrete-time dynamic program defined over the time slots $0, \ldots, K-1$. Let T_z^k denote the indoor temperature at the beginning of time slot k. We use T_z^k for the system state in defining the dynamic programming recursion. Let $C_k(T_z^k)$ denote the optimal cost-to-go (i.e., the optimal cost as defined by (1) between time $k\tau$ and $K\tau = W$) when the system is at state T_z^k at time $t = k\tau$. Define $C_K(T_z^K) = 0$.

Proposition 1. *For any time slot k, let the electricity price $p(t)$ and the ambient temperature $T_\infty(t)$ remain constant at p_k and T_∞^k, respectively, i.e., $p(t) = p_k$ and $T_\infty(t) = T_\infty^k$ for $t \in [k\tau, (k+1)\tau)$, $k = 0, \ldots, K-1$. Then the optimal cost of the problem defined in (1)-(5) is obtained as $C_0(T_z^0)$, which can be computed using the backward induction equation,*

$$C_k(T_z^k) = \min_{T_z^{k+1}} \left\{ p_k c_k(T_z^k \rightarrow T_z^{k+1}) + C_{k+1}(T_z^{k+1}) \right\}, \quad (6)$$

where $c_k(T_z^k \rightarrow T_z^{k+1})$ denotes the minimum electrical energy required for state transition from T_z^k to T_z^{k+1} over slot k.

Proof. (Outline) Denote $u_k = \{u(t), t \in [k\tau, (k+1)\tau]\}$. This, along with the fact that $p(t) = p_k,\ k\tau \leq t \leq (k+1)\tau$ reduces the optimization objective function in (1) to,

$$\mathcal{J}(u_k) = \frac{1}{\eta} \sum_{k=0}^{K-1} p_k f(u_k). \quad (7)$$

where $f(u_k) = \int_{t=k\tau}^{(k+1)\tau} u(t)\, dt$. Also, from solving the first order ODE in (2) and expressing T_z^{k+1} in terms of T_z^k and u_k, we can write,

$$T_z^{k+1} = T_\infty^k - Ru_k + (T_z^k - T_\infty^k + Ru_k)\exp(-\tau/RC). \quad (8)$$

Equations (7) and (8), along with equations (3)-(5) allows us to express the original problem as a discrete time optimal control problem in its canonical form where T_z^k is the system state and u_k is the control applied at time $k\tau$. We can then solve this by applying dynamic programming, where the optimal cost is obtained as $C_0(T_z^0)$, which in turn, can be computed using the backward induction equation [16],

$$C_k(T_z^k) = \min_{u_k} \left\{ p_k c_k(T_z^k, u_k) + C_{k+1}(g(T_z^k, u_k)) \right\}, \quad (9)$$

where $c_k(T_z^k, u_k)$ denotes the one stage cost-to-go from an initial temperature T_z^k and when the control signal applied is u_k. The temperature T_z^{k+1} at $(k+1)\tau$ is given as $T_z^{k+1} = g(T_z^k, u_k)$ where explicit expression of $g(T_z^k, u_k)$ can be found from (8). Note that we can rearrange (9) to write,

$$C_k(T_z^k) = \min_{T_z^{k+1}} \left\{ \min_{u_k | T_z^{k+1}} \left\{ p_k c_k(T_z^k, u_k) + C_{k+1}(g(T_z^k, u_k)) \right\} \right\}, \quad (10)$$

where the inner minimization ($\min_{u_k | T_z^{k+1}}$) is performed over all u_k such that the system attains a given system state T_z^{k+1} at the end of time slot k (i.e. start of time slot $k+1$). Noting that in (10), $\min_{u_k | T_z^{k+1}} \left\{ p_k c_k(T_z^k, u_k) \right\} = p_k c_k(T_z^k \rightarrow T_z^{k+1})$ and $\min_{u_k | T_z^{k+1}} C_{k+1}(g(T_z^k, u_k)) = \min_{u_k | T_z^{k+1}} C_{k+1}(T_z^{k+1}) = C_{k+1}(T_z^{k+1})$, the result in (6) follows. □

In general, the minimization in (6) is taken over the feasible indoor temperature range $[T_z^{lb}, T_z^{des}]$. However, note that parameters such as $\tau, T_\infty^k, \bar{u}$ determine whether state T_z^{k+1} is reachable from T_z^k over time slot k. In Section IV-B, we provide an explicit condition that determines this reachability; if T_z^{k+1} is not reachable from T_z^k, then $c_k(T_z^k \rightarrow T_z^{k+1})$ is assumed to be ∞. Please see Figure 1 for illustration of the dynamic programming recursion.

To evaluate the complexity of the dynamic program as described in (6), we assume that the indoor temperature has been discretized to m levels per °C. Further, let the time complexity of computing $c_k(T_z^k \rightarrow T_z^{k+1})$ be α; we show in Section IV-B that α is $O(1)$. Then for a given T_z^k, the time complexity of obtaining the corresponding optimal T_z^{k+1} (finding the minimum among $m\Delta$ possibilities) is given by $\alpha(m\Delta)$. Therefore, the time complexity of obtaining $C_k(T_z^k)$ for all $m\Delta$ possibilities of T_z^k becomes $\alpha(m\Delta) \times (m\Delta) = \alpha(m\Delta)^2$. The time complexity of computing the optimal cost, $C_0(T_z^0)$, which is obtained through backward induction from $k = K-1$ to $k = 0$, can thus be expressed as $\alpha(m\Delta)^2 \times K = \alpha(m\Delta)^2 \frac{W}{\tau}$.

B. Optimal State Transition Trajectory

In this section, we provide a set of results that show how $c_k(T_z^k \rightarrow T_z^{k+1})$, the optimal electrical energy of transitioning from state T_z^k to state T_z^{k+1} in time slot k, can be computed in constant time. Let $u_k^*(T_z^k \rightarrow T_z^{k+1}) = \Big\{u^*(t), t \in [k\tau, (k+1)\tau),$ such that $T_z(k\tau) = T_z^k$ and $T_z((k+1)\tau) = T_z^{k+1}\Big\}$ denote the optimal solution that achieves the minimum energy for this state transition. The following result provides a characterization of the structure of $u_k^*(T_z^k \rightarrow T_z^{k+1})$.

Proposition 2. *There exist time durations $\tau_1, \tau_2, \tau_3 \geq 0$, satisfying $\tau_1 + \tau_2 + \tau_3 = \tau$, such that*

$$u_k^*(T_z^k \rightarrow T_z^{k+1}) = \begin{cases} 0, & k\tau \leq t < k\tau + \tau_1, \\ u_{\text{avg}}^{des}, & k\tau + \tau_1 \leq t < k\tau + \tau_1 + \tau_2, \\ \bar{u}, & k\tau + \tau_1 + \tau_2 \leq t < (k+1)\tau. \end{cases}$$

Furthermore, $T_z(t) = T_z^{des}$ for $k\tau + \tau_1 \leq t < k\tau + \tau_1 + \tau_2$.

Proof. (Outline) Note that from (2), we can express the total cooling energy consumed by an HVAC while transitioning from a certain indoor zone temperature T_z^k (at time $t = k\tau$) to a feasible T_z^{k+1} (at time $t = (k+1)\tau$) as,

$$E(T_z^k \rightarrow T_z^{k+1}) = \int_{t=k\tau}^{(k+1)\tau} u \, dt = \frac{\Psi}{R} - \underbrace{C(T_z^{k+1} - T_z^k)}_{\spadesuit}, \tag{11}$$

where $\Psi = \int_{t=k\tau}^{(k+1)\tau} \big(T_\infty^k - T_z(t)\big) \, dt$ is the difference between the area under curve of the ambient temperature T_∞^k and that of the indoor zone temperature $T_z(t)$ during the time interval $[k\tau, (k+1)\tau]$, and is assumed to be ≥ 0 for our purposes ($T_\infty \geq T_z(t), \ \forall t$). Note that for a given T_z^k and T_z^{k+1}, the term $\spadesuit$ is a constant and thus, an energy optimal trajectory of transition from a given T_z^k to a given T_z^{k+1} has the least feasible value of Ψ and hence, the *highest* feasible trajectory of $T_z(t)$.

For the sake of contradiction, assume that there exists some other control policy $\hat{u}_k(T_z^k \rightarrow T_z^{k+1})$ that results in lesser energy consumption when compared to $u_k^*(T_z^k \rightarrow T_z^{k+1})$. By our preceding argument, the trajectory for $\hat{u}_k(T_z^k \rightarrow T_z^{k+1})$ must be "higher" than the trajectory for $u_k^*(T_z^k \rightarrow T_z^{k+1})$ for some finite duration in $[k\tau, (k+1)\tau]$. Clearly, this cannot happen in $[k\tau + \tau_1, k\tau + \tau_1 + \tau_2]$ since that would imply that the upper bound of comfort T_z^{des} is violated. Also, it is easy to see that in $[k\tau, k\tau + \tau_1]$, since the initial temperature T_z^k is given, one can only raise the trajectory of $T_z(t)$ above the one achieved with $u_k^*(T_z^k \rightarrow T_z^{k+1})$ by adding additional heat into the system, which is not possible without violating system dynamics. Now, let there be a time $t_0 \in [k\tau + \tau_1 + \tau_2, (k+1)\tau]$ such that the temperature at t_0 for $\hat{u}_k$ ($\hat{T}_z(t_0)$) is higher than that for u_k^* ($T_z^*(t_0)$) i.e. $\hat{T}_z(t_0) > T_z^*(t_0)$. Then, it is easy to see that it is only possible to bring the temperature down from $\hat{T}_z(t_0)$ at time t_0 to T_z^{k+1} at time $(k+1)\tau$ with a cooling power larger than $\bar{u}$. Since that violates system dynamics, this scenario is also not possible. Thus, our assumption is incorrect and the trajectory for $\hat{u}_k(T_z^k \rightarrow T_z^{k+1})$ cannot be higher than the trajectory for $u_k^*(T_z^k \rightarrow T_z^{k+1})$ in $[k\tau, (k+1)\tau]$. The result follows. □

Intuitively, Proposition 2 implies that during any time slot (over which the electricity price and ambient temperature are assumed constant), the AC OFF-time (τ_1) can occur only at the beginning of the time slot, and the AC ON-time (τ_3) can occur only at the end of the time slot. Furthermore, for the rest of the time in that time slot, the AC must operate so as to maintain the indoor temperature at the desired temperature, T_z^{des}.

Proposition 2 greatly reduces the set of possibilities of the optimal state trajectory (of the indoor temperature $T_z(t)$) between T_z^k and T_z^{k+1}. Note that some of durations τ_1, τ_2, τ_3, could possibly be zero. Figure 2 illustrates two of the possibilities of the optimal trajectory of $T_z(t)$ over the time slot k ($\tau_1, \tau_3 > 0$, while $\tau_2 > 0$ in Case (i) and $\tau_2 = 0$ in Case (ii)). Note that Proposition 2 implies that for given T_z^k, T_z^{k+1}, the optimal control $u_k^*(T_z^k \rightarrow T_z^{k+1})$ in time slot k is completely determined by the OFF-time τ_1 and the ON-time, τ_3 (note that $\tau_2 = \tau - \tau_1 - \tau_3$). Corollary 1, which is stated below, shows how these time durations (τ_1, τ_2, τ_3) and the optimal energy $c_k(T_z^k \rightarrow T_z^{k+1})$ can be computed efficiently.

Before stating the result, we define two quantities $T_z^{(-)}, T_z^{(+)}$ that represent the lower and upper bound on the set of states reachable at time $t = (k+1)\tau$ starting from T_z^k at time $t = k\tau$.

$$T_z^{(+)} = T_\infty^k + (T_z^k - T_\infty^k)\exp(-\tau/RC), \tag{12}$$
$$T_z^{(-)} = T_\infty^k - R\bar{u} + (T_z^k - T_\infty^k + R\bar{u})\exp(-\tau/RC). \tag{13}$$

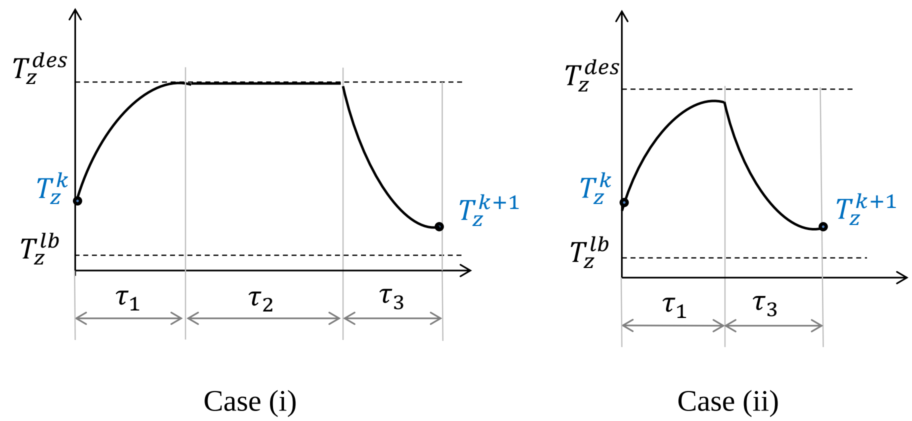

Fig. 2. Trajectory of optimal state (indoor temperature) transition within a time slot (period of constant electricity price and ambient temperature).

Corollary 1. *The optimal electrical energy required for state transition* $c_k(T_z^k \to T_z^{k+1}) = \infty$ *if* $T_z^{k+1} < T_z^{(-)}$ *or* $T_z^{k+1} > T_z^{(+)}$*. Otherwise, it is given by*

$$c_k(T_z^k \to T_z^{k+1}) = \Big(\frac{T_\infty^k - T_z^{des}}{R\eta}\Big)\tau_2 + \Big(\frac{\bar{u}}{\eta}\Big)\tau_3, \qquad (14)$$

where τ_2, τ_3 *(and* $\tau_1 = \tau - \tau_2 - \tau_3$*) are given by Cases (i) and (ii) below:*
Case (i): If $\tau \geq -RC \log\Big(\frac{T_z^{des} - T_\infty^k}{T_z^k - T_\infty^k}\Big)\Big(\frac{T_z^{k+1} - T_\infty^k + R\bar{u}}{T_z^{des} - T_\infty^k + R\bar{u}}\Big)$,

$$\begin{aligned}\tau_1 &= -RC \log\Big(\frac{T_z^{des} - T_\infty^k}{T_z^k - T_\infty^k}\Big),\\ \tau_3 &= -RC \log\Big(\frac{T_z^{k+1} - T_\infty^k + R\bar{u}}{T_z^{des} - T_\infty^k + R\bar{u}}\Big),\\ \tau_2 &= \tau - \tau_1 - \tau_3;\end{aligned}$$

Case (ii): If $\tau < -RC \log\Big(\frac{T_z^{des} - T_\infty^k}{T_z^k - T_\infty^k}\Big)\Big(\frac{T_z^{k+1} - T_\infty^k + R\bar{u}}{T_z^{des} - T_\infty^k + R\bar{u}}\Big)$,

$$\begin{aligned}\tau_1 &= -RC \log\Big(\frac{\phi R\bar{u}}{T_z^k - T_\infty^k}\Big),\\ \tau_3 &= -RC \log\Big(\frac{T_z^{k+1} - T_\infty^k + R\bar{u}}{\phi R\bar{u} + R\bar{u}}\Big),\\ \tau_2 &= 0,\end{aligned}$$

where $\phi = \frac{\exp(-\tau/RC)\Big(\frac{T_z^k - T_\infty^k}{T_z^{k+1} - T_\infty^k + R\bar{u}}\Big)}{1 - \exp(-\tau/RC)\Big(\frac{T_z^k - T_\infty^k}{T_z^{k+1} - T_\infty^k + R\bar{u}}\Big)}$.

Proof. (Outline) It is easy to see that $T_z^{(+)}$ and $T_z^{(-)}$ denote the upper and the lower bound of reachable states from T_z^k, given a T_∞^k and a fixed τ. Any T_z^{k+1} such that $T_z^{k+1} > T_z^{(+)}$ or $T_z^{k+1} < T_z^{(-)}$ is not attainable and thus the cooling energy needed in such a case is undefined.

From Proposition 2, it is clear that the initial duration $[k\tau, k\tau + \tau_1]$ is the period where HVAC is OFF. Assume that the indoor zone temperature reaches T_z^{des} in this duration. Now, solving (2) subject to $u(t) = 0$, $T_z(0) = T_z^k$ and $T_z(\tau_1) = T_z^{des}$, we get,

$$\tau_1 = -RC \log\Big(\frac{T_z^{des} - T_\infty^k}{T_z^k - T_\infty^k}\Big). \qquad (15)$$

Similarly, for computing τ_3, which is the time taken for the HVAC to cool (using cooling power $\bar{u}$) from T_z^{des} to T_z^{k+1} we can solve (2) subject to $T_z(0) = T_z^{des}$, $u(t) = \bar{u}$ and $T_z(\tau_3) = T_z^{k+1}$, and can write,

$$\tau_3 = -RC \log\Big(\frac{T_z^{k+1} - T_\infty^k + R\bar{u}}{T_z^{des} - T_\infty^k + R\bar{u}}\Big). \qquad (16)$$

Now, if $\tau \geq \tau_1 + \tau_3$ i.e. $\tau \geq -RC \log\Big(\frac{T_z^{des} - T_\infty^k}{T_z^k - T_\infty^k}\Big)\Big(\frac{T_z^{k+1} - T_\infty^k + R\bar{u}}{T_z^{des} - T_\infty^k + R\bar{u}}\Big)$ i.e. the conditions corresponding to Case (i), we can directly get τ_1 and τ_3 as computed in (15) and (16). Moreover, we can directly compute τ_2 as $\tau_2 = \tau - \tau_1 - \tau_3$.

For the other case i.e. Case (ii), $\tau < \tau_1 + \tau_3$ which can be written as, $\tau < -RC \log\Big(\frac{T_z^{des} - T_\infty^k}{T_z^k - T_\infty^k}\Big)\Big(\frac{T_z^{k+1} - T_\infty^k + R\bar{u}}{T_z^{des} - T_\infty^k + R\bar{u}}\Big)$. This corresponds to the case where τ is not large enough for the $T_z(t)$ to reach T_z^{des} while transitioning to T_z^{k+1}. Thus, there exists an intermediate temperature T_z^{int} such that $T_z^{lb} \leq T_z^{int} \leq T_z^{des}$. Note that τ_1 now gives the time taken for $T_z(t)$ to rise from T_z^k to T_z^{int} (while HVAC is OFF) and τ_3 gives the time taken for $T_z(t)$ to transition from T_z^{int} to T_z^{k+1} (while the HVAC is ON). It is also easy to see that $\tau_1 + \tau_3 = \tau$ and hence, $\tau_2 = 0$. Now, τ_1 can be computed as earlier (for Case (i)) by solving for (2), subject to $u(t) = 0$, $T_z(0) = T_z^k$, $T_z(\tau_1) = T_z^{int}$ and can be written as,

$$\tau_1 = -RC \log\Big(\frac{T_z^{int} - T_\infty^k}{T_z^k - T_\infty^k}\Big). \qquad (17)$$

Similarly, τ_3 can be found by solving (2) subject to $u(t) = \bar{u}$, $T_z(0) = T_z^{int}$ and $T_z(\tau_3) = T_z^{k+1}$ as,

$$\tau_3 = -RC \log\Big(\frac{T_z^{k+1} - T_\infty^k + R\bar{u}}{T_z^{int} - T_\infty^k + R\bar{u}}\Big). \qquad (18)$$

Since $\tau_1 + \tau_3 = \tau$, with some algebraic rearrangement, we can express T_z^{int} as,

$$T_z^{int} = T_\infty^k + \frac{\exp(-\frac{\tau}{RC} - \log\frac{T_z^{k+1} - T_\infty^k + R\bar{u}}{T_z^k - T_\infty^k})}{1 - \exp(-\frac{\tau}{RC} - \log\frac{T_z^{k+1} - T_\infty^k + R\bar{u}}{T_z^k - T_\infty^k})} R\bar{u}. \qquad (19)$$

Replacing T_z^{int} as found in (19) in (17), (18) and subsequent algebraic manipulation yields the expressions for τ_1 and τ_3 as seen in Case (ii) for Corollary 1.

Noting that during $[k\tau+\tau_1, k\tau+\tau_1+\tau_2]$ and $[k\tau+\tau_1+\tau_2, (k+1)\tau]$, the electrical power consumed is $\frac{T_\infty^k - T_z^{des}}{R\eta}$ and $\frac{\bar{u}}{\eta}$ respectively, the expression for $c_k(T_z^k \to T_z^{k+1})$ can be computed as given in (14). □

Since the expressions are all in closed form, Corollary 1 can be used to compute $c_k(T_z^k \to T_z^{k+1})$ in constant time. Refer to Figure 2 again for an illustration of Cases (i) and (ii) of Corollary 1.

V. Numerical Study

We consider a cooling space having $R = 6.67\ ^\circ\text{C/kW}$ and $C = 2 \times 10^3\ \text{kJ}/^\circ\text{C}$ in New York City. Also, we assume that $\bar{u} = 6$ kW and the coefficient of performance (COP) $\eta = 2.0$. We now study the effect of our proposed algorithm under a set of different scenarios to understand its efficacy. Also, assume $T_z^{des} = 22^\circ\text{C}$ and $T_z^{ub} = 20^\circ\text{C}$ i.e. the comfort range $\Delta = 2^\circ\text{C}$. We first consider the case with day-ahead market prices for a given day (July 18, 2012).

From Figure 3, we can see that in our selected window, the price starts to peak around mid-day (12 pm) and persists till the afternoon (5 pm). The ambient temperature is also seen to be around $30 - 35^\circ\text{C}$ (high) during this period. From Figure 3, we observe that our algorithm successfully switches OFF the HVAC in the periods where prices are high, and *precools* to within the comfort limits in the preceding time slots of relatively less energy price. We compare the performance with respect to a baseline control strategy where the indoor zone temperature is always maintained at T_z^{des}. For this particular day of operation, we find a cost savings of approximately 6% with precooling.

Next, we do a similar study with real-time prices (assume that a perfect prediction of such prices are available for our selected look-ahead window i.e. July 1, 2012 – 2pm to 4pm). As in the earlier case, in Figure 4, we show how the dynamic program is able to optimize price of energy usage by causing the HVAC to precool in the less expensive time slots and keeping it OFF during the relatively more expensive time slots. In this 2-hour window under consideration, our algorithm obtains a cost savings of approximately 21% when compared to a similar baseline strategy as before (HVAC maintains T_z^{des} throughout the look-ahead window). The high savings can be attributed to completely switching OFF the HVAC without violation of comfort bounds in very expensive time slots (time slots 10 to 15 as shown in Figure 4). Note that in general, day-ahead prices show less volatile behavior than real-time prices; hence the savings obtained for day-ahead prices is typically lesser compared to that for real-time prices.

We then run our algorithm for the entire month of July 2012 (peak summer) under day-ahead pricing assumptions. The length of the look-ahead window for each dynamic program in this case is taken to be a day (24 hours). We observe that our precooling strategy obtains a savings of approximately 2% which is less as compared to Texas [12]. This is because, as compared to Texas, the average values and the volatility of day-ahead prices in a cooler state like New York is considerably lower, which implies that the cost savings through precooling under exposure to such prices is lesser. We now repeat our experiment with real-time prices. The length of the look-ahead window for each dynamic program in this case is taken to be 1 hour. We now observe that with real time prices the net monthly savings made possible are approximately 15% and 7% for July 2012 (the warmest summer month in New York) and September 2012 (a fringe summer month), respectively. Thus, we can infer that under exposure to real-time prices, our proposed framework can help the LSE (aggregator) to obtain significant savings in energy costs both in both peak and fringe summer months.

VI. Concluding Remarks

In this paper, we considered the optimal precooling problem in buildings under time-varying electricity prices and posed it as an optimal control problem. We provided a structural property of the minimum-cost state trajectory between two given temperature set-points under fixed electricity price and ambient temperature. We utilized this structure, and the piece-wise constant nature of the time-varying electricity prices, to show how dynamic programming can be utilized to solve the optimal control problem exactly in low time complexity. We utilized this solution approach on some data sets collected over summer months in New York to study the benefit of precooling under both day-ahead and real-time electricity prices. Our results show that the cost savings obtained by precooling under day-ahead prices (which are known in advance) is quite modest. While the potential for cost savings under real-time prices is quite large, it is worth noting that real-time prices can be difficult to predict. The question of what fraction of this cost savings can be attained with practical real-time price prediction strategies remains open for future investigation.

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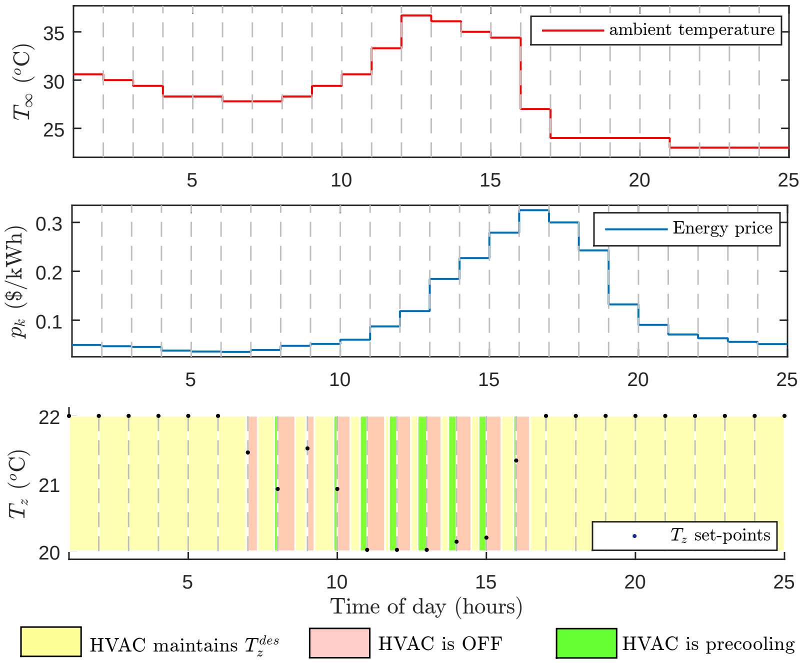

Fig. 3. Optimal solution computed using our dynamic programming approach along with ambient temperature and day-ahead electricity prices for July 18, 2012. Time duration for which HVAC is OFF, HVAC is maintaining upper bound and HVAC is precooling are also shown. The dots represent T_z at the time slot boundaries.

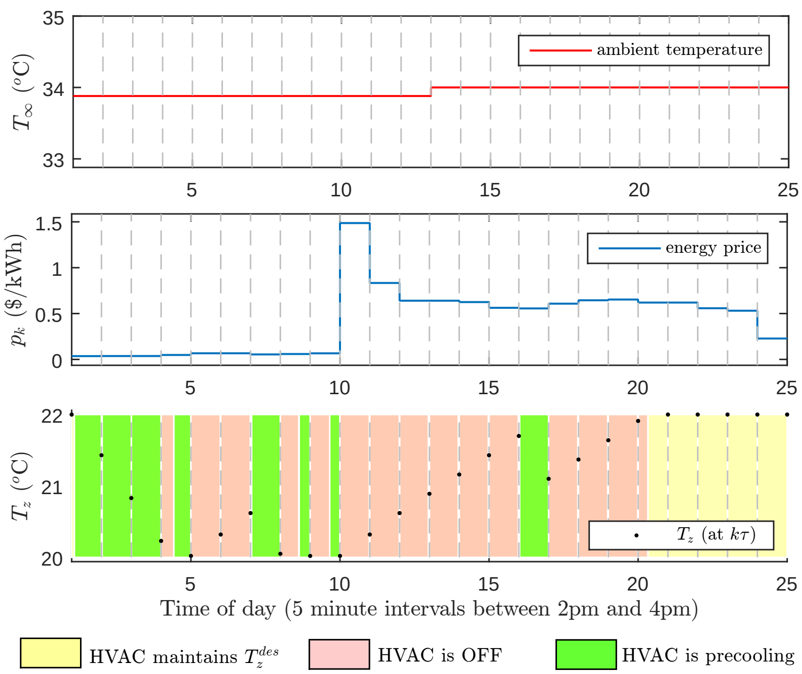

Fig. 4. Optimal solution computed using our dynamic programming approach along with ambient temperature and real-time electricity prices for July 1, 2012 (2pm - 4pm). Time duration for which HVAC is OFF, HVAC is maintaining upper bound and HVAC is precooling are also shown. The dots represent T_z at the time slot boundaries..

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