Optimization of Windfarm Power by Controlling the Yaw Angle using Dynamic Programming

Zamiyad Dar, Koushik Kar, Onkar Sahni and Joe H. Chow

***Abstract*—In this paper we extend the Park and Jensen wake model to include the effects of yaw misalignment and wake deflection of wind turbines. We perform a numerical study to find the optimal values of induction factor and yaw misalignment angle of wind turbines in a single row of a windfarm for achieving the maximum total power with wake effects. Our study shows that the maximum power is achieved by keeping the induction factor close to $\frac{1}{3}$ and only changing the yaw angle to deflect the wake. We then propose a dynamic programming framework (DPF) to maximize the total power production of a windfarm using yaw angle as the control variable. We compare the total windfarm power achieved with our DPF with the total power values obtained through greedy control strategy and induction factor optimization.**

***Index Terms*—Wind energy, Windfarm power optimization, Yaw angle control, Dynamic programming.**

I. Introduction

THE amount of installed wind energy in power systems is increasing at a steady rate. In the United States, the installed wind power capacity increased by 28% in 2012 [1]. The US Department of Energy has set a target to achieve 20% wind energy penetration by 2030 [2]. Currently, windfarms are operated in such a way that each turbine is operated at its individual optimum operation point according to the Beltz limit [3]. When a wind turbine extracts energy from the wind, the speed of wind is reduced behind the turbine and the downstream turbines receive a reduced wind speed. In a large windfarm, where the turbines are arranged in a row, the downstream turbines are always operating in the wake of the upstream wind turbines. This results in reduced power production from downstream wind turbines. The operation of each turbine at its optimum level may not result in the maximum total power from the windfarm [4]. According to [5], onshore windfarms in the US have been generating 10% to 15% less energy than their maximum capacity.

In order to analyze the cumulative power of the windfarm, it is necessary to consider the wake effect of upstream turbines on the downstream turbines. The most commonly used wake model to study the wake effects in a windfarm is the Park and Jensen model [6]. It has been observed both theoretically [3] and experimentally [7] that rotating the nacelle of a wind turbine changes the direction of the wake behind the wind turbine. The two factors that can be used to change the power production from individual turbines are induction factor and yaw misalignment. Induction factor is a measure of the difference between the wind velocity in front of the turbine and behind the turbine [3]. It can be controlled by changing the generator torque or blade pitch [3]. The yaw misalignment angle or yaw angle is the angle between the axis of the rotor plane and the incoming wind. If the wind turbine is operated in a way that the rotor plane faces the wind perpendicularly then the yaw offset angle is zero degrees.

Previous research on optimization of power production in a windfarm has focused on both induction factor and yaw angle. In [8] and [9], the total power of a windfarm is optimized by finding the optimal induction factors of individual turbines. In [10], the optimal yaw angle and blade pitch of individual turbines are determined to maximize the total power in a windfarm using a data mining approach. The yaw settings of individual turbines are optimized to maximize the total windfarm power in [11]. [12] shows the optimal yaw angle pattern for a windfarm consisting of three turbines. Experimental results in [13] and [7] show that the total windfarm power can be increased by changing the yaw angles of different turbines. In [14] and [15] both yaw angle and induction factor of each turbine are optimized to achieve the maximum total power. Dynamic programming is used in [16] and [17] to find the optimal set of induction factors for individual turbines to maximize the total windfarm power.

Our current research extends this subject and adds two novel contributions to the existing literature. Firstly, we establish through numerical simulations that the optimum power could be achieved by only controlling the yaw angle of the wind turbines and keeping the induction factors approximately equal to the Beltz limit of $\frac{1}{3}$. This reduces the number of variables in the windfarm power maximization problem by a factor of two. The dynamic programming framework (DPF) described in [18] has only been previously employed for controlling the induction factor to maximize the total windfarm power. Our second novel contribution is the formulation of a dynamic programming technique for controlling the yaw angle of individual wind turbines to maximize the total wind farm power.

In this paper, we analyze the simultaneous effects of induction factor and yaw angle on the power and wake of wind turbines. In Section II, we consider a windfarm consisting of a single row of wind turbines and formulate the equations for calculation of total power of that windfarm accounting for the wake effects and yaw misalignment. In Section III, we show the results of numerical simulations to find the optimal values of induction factor and yaw angle of each turbine in a row of turbines in a windfarm which maximize the total

Zamiyad Dar, Koushik Kar, Onkar Sahni and Joe H. Chow are with Rensselaer Polytechnic Institute, Troy, NY, 12180. Their emails are {darz, kark, chowj, sahni}@rpi.edu

Manuscript received July 19, 2015; revised September 17, 2015.

TABLE I
VARIABLE NOTATION AND EXPLANATION

Notation	Explanation
r_0	Radius of the wind turbine
k	Wake decay constant
S	Horizontal spacing between two wind turbines in the windfarm (Fig. 3)
a_i	Induction factor of the i^{th} wind turbine
ϕ_i	Angle that the wake makes with the rotor's axis of rotation behind the turbine i
θ_i	Yaw misalignment angle of the i^{th} wind turbine
L	Spacing between wind turbines divided by the diameter of rotor of the turbine: $L = \frac{S}{2r_0}$
ρ	Density of air
A	Area swept by the rotor of the wind turbine. $A = \pi r_0^2$
U_i	Velocity faced by the i^{th} wind turbine
g_i	Power generated by the i^{th} wind turbine
P_{Wake}	Maximum power of windfarm with the wake effects
P_{NoWake}	Maximum power of windfarm without considering any wake effects
η_{wf}	Efficiency of the windfarm
N	Total number of turbines in the windfarm
WT_i	i^{th} wind turbine in the wind farm

windfarm power. In Section IV, we present the DPF for the optimization of total power in a windfarm using yaw angle as the control variable and keeping the induction factor fixed at a value of $\frac{1}{3}$. In Section V, we show how the optimal power using our approach compares with the other existing approaches for the maximization of total power in windfarms. Section VI concludes and summarizes the paper.

II. Wake Model

The notations of variables used in this paper are provided in Table I.

In order to analyze the total power of the windfarm, it is necessary to consider the wake effects of upstream turbines on the downstream turbines. There are different far wake models present in literature which describe the interaction between turbines in a windfarm [12]. Park and Jensen model [6] is one of the most commonly used far wake model for wind turbines. Near wake is the wake region approximately one to three rotor diameters downstream of a wind turbine and far wake region is the wake region more than three rotor diameters behind a wind turbine [19]. In [6], the authors give the following expression for the velocity of wind at a point at horizontal distance x and radial distance r from the center of wake behind a wind turbine in the far wake region as shown in Fig. 1.

$$U(x) = \begin{cases} U\left[1 - 2a\left(\frac{r_0}{r_0+kx}\right)^2\right] & \text{if } r \leq r_0 + kx, \\ U & otherwise \end{cases} \quad (1)$$

The above expression implies that the wake effect is constant throughout the area included in the affected radius and zero outside that radius. According to [6], the radius of the wake behind a wind turbine increases linearly with the distance from the wind turbine and is given as $r_0 + kx$. The constant k is the wake decay constant and dictates the expansion of wake behind a wind turbine. Usually k is taken to be 0.075 for

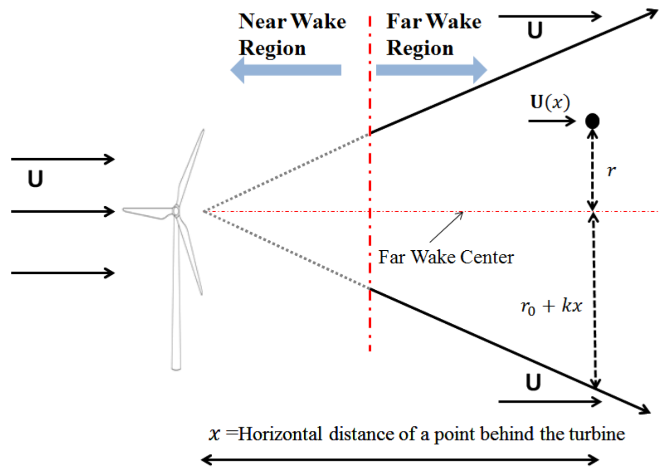


Fig. 1. Near and far wake regions behind a wind turbine facing wind at a speed of U units. Equation (1) calculates the wind velocity $U(x)$ at a point at horizontal distance x and radial distance r behind the turbine

onshore turbines [20] and [21]. In order to have a wake model for far wake region in which the wind velocity changes more gradually in the radial direction, the authors in [6] introduce a modulating factor of $\frac{(1 + \cos 9\theta)}{2}$. Using this modulating factor in (1), the modified Park and Jensen model is given in (2) and shown in Fig. 2.

$$U(x,\theta) = \begin{cases} U\left[1 - 2a\left(\frac{r_0}{r_0+kx}\right)^2 \frac{1+\cos(9\theta)}{2}\right] & \theta < 20^\circ, \\ U & \theta \geq 20^\circ \end{cases} \quad (2)$$

Authors in [6] show that practical measurements match this expression and the deficit in velocity is negligible once the angle of wake from the center of upstream turbine is increased beyond 20 degrees.

A. Wake deflection due to yaw misalignment

Wind turbines are typically operated in a way such that they face the wind perpendicular to the rotor plane. This is a state of zero yaw misalignment or yaw angle. Any rotation of the wind turbine from this position would change the yaw angle. Changing the yaw angle from its usual zero degree position deflects the wake behind the wind turbine as shown in Fig. 3. The direction of wind turbine rotation and wake deflection shown in Fig. 3 is consistent with [3], [11] and [14]. This effect has also been observed in [13] and [7] through various experiments. It is proposed in [3] and [7] that the deflection of wake is directly proportional to the magnitude of the yaw angle. An expression is proposed in [3] which relates the deflection of wake behind the wind turbine to its yaw angle as follows.

$$\phi_{i+1} = (0.6a_{i+1} + 1) \times \theta_{i+1} \quad (3)$$

Fig. 3 shows two wind turbines with a horizontal seperation of S units. The yaw angle of the upstream wind turbine (WT_{i+1}) is θ_{i+1}. This deflects the wake behind the upstream turbine in accordance with (3). The center of the wake is deflected away from the horizontal line at an angle of ϕ_{i+1} and

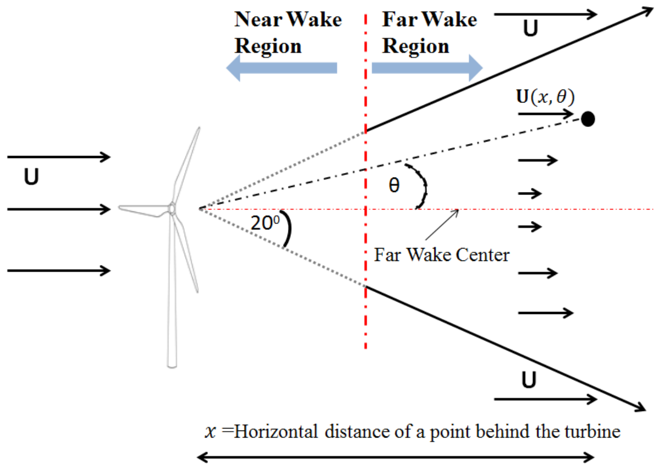


Fig. 2. Near and far wake regions behind a wind turbine facing wind at a speed of U units. Equation (2) calculates the wind velocity $U(x, \theta)$ at a point at horizontal distance x and at angle θ behind the turbine

this changes the effective distance for wake effects between the two turbines to $S\cos(\phi_{i+1})$ as shown in Fig. 3.

The experimental results from [22] show that the tangential component of the wind velocity produced due to the wake effects of an upstream turbine is less than 5% immediately behind the turbine and almost negligible as we move farther than 3 rotor diameter downstream. We assume that the tangential or crosswind component of the velocity produced due to the wake effect of the upstream turbine is negligible at the downstream turbine and can be ignored. We also assume that the wind shear or gradient in wind speed incident on the turbine blades is also negligible. With these assumptions and combining (2) and (3), we can modify the Park and Jensen model from [6] to include the effects of yaw angle and get the velocity at the rotor of turbine i as

$$U_i = \begin{cases} U_{i+1}[1 - 2a_{i+1}\left(\frac{1}{1+2kL\cos\phi_{i+1}}\right)^2 & \\ \times\cos^2(4.5\,\phi_{i+1})] & \phi_{i+1} < 20^\circ, \\ U_{i+1} & \phi_{i+1} \geq 20^\circ, \end{cases} \quad (4)$$

and $L = \dfrac{S}{2\,r_0}$ is a spacing factor which gives the distance between turbines as a multiple of turbine diameter.

B. Effect of yaw angle on wind turbine power

A single wind turbine produces its maximum power when the rotor is facing the wind perpendicularly and yaw angle is zero degrees. Changing the yaw angle of a wind turbine from zero degrees reduces the power of that individual wind turbine. There are different expressions present in literature for the effect of yaw angle on the power of a wind turbine. In [13], it was shown that the wind power of a single turbine is related to yaw angle through a factor of $\cos^n\theta_i$, where n is recommended to be between 1.5 and 2. In [23], the value of n is approximately 2. In [24] and [25], a factor of $\cos^3\theta_i$ is suggested while [14] and [15] suggest a factor of $\cos^2\theta_i$. We observe that all these factors are within an average range of 2% for yaw angles of less than 20 degrees. We select a

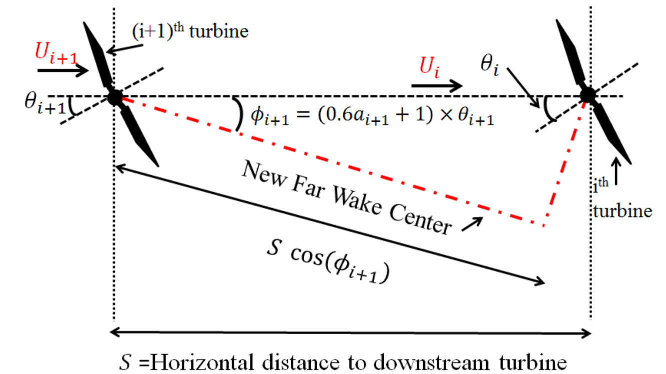


Fig. 3. Two wind turbines spaced S units apart. The yaw angle of the upstream $i+1$ turbine is θ_{i+1} and the wake is deflected at an angle ϕ_{i+1}

factor of $\cos^2\theta_i$. With U_i taken from (4), the power of the i^{th} turbine in Fig. 3 will be

$$g_i = \frac{1}{2}\rho A(4a_i(1-a_i)^2) \times U_i^3 \times \cos^2\theta_i. \quad (5)$$

III. Numerical Results

We proceed with this modified Park and Jensen model containing the effect of yaw angle on wake and velocity of wind from (4), and the effect of yaw angle on power of a wind turbine from (5). We used (4) and (5) to perform simulations in Matlab to obtain the total power of a windfarm. We considered four different windfarms consisting of 2 – 5 wind turbines arranged in a row i.e, $N \in \{2, 3, 4, 5\}$. We calculate the velocity at the i^{th} turbine using (4) and then calculate the power of the i^{th} turbine using (5). We calculate total power of these four windfarms at different values of induction factor and yaw angles. For each wind turbine, we vary the value of induction factor from 0 to $\frac{1}{3}$ in increments of 10^{-5}, and value of yaw angle from 0 to 20 degrees for each value of induction factor. We consider three values of spacing between turbines in each windfarm. i.e,

$$L = \frac{x}{2\,r_0} \in \{5, 10, 15\}$$

A. Maximum power values

For each of these spacing, we obtain induction factor and yaw angle values for all wind turbines in a windfarm at maximum total power. We calculate the wind farm efficiency in the same manner as [25]. The windfarm efficiency is calculated from the total maximum power produced in the presence of wake effects and the total maximum power produced without considering any wake effects.

$$\eta_{wf} = \frac{P_{Wake}}{P_{NoWake}} \times 100\% \quad (6)$$

The results for $L = 5$, $L = 10$ and $L = 15$ are shown in Table II, III and IV respectively. We observe that increasing the spacing between the wind turbines increases the optimal power production. This is intuitively expected because the wake effect is reduced as we move farther behind the turbine. An

TABLE II
NUMERICAL SIMULATIONS FOR SPACING OF 5 ROTOR DIAMETERS

Parameter	Turbine 1	Turbine 2	Turbine 3	Turbine 4	Turbine 5	Windfarm efficiency
OIF	$\frac{1}{3}$	$\frac{1}{3}$	.	.	.	96.07%
Yaw angle (degrees)	15.91	0	.	.	.	
OIF	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.	.	94.70%
Yaw angle (degrees)	16.26	15.91	0	.	.	
OIF	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.	94.00%
Yaw angle (degrees)	16.39	16.26	15.91	0	.	
OIF	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	93.58%
Yaw angle (degrees)	16.46	16.39	16.26	15.91	0	

TABLE III
NUMERICAL SIMULATIONS FOR SPACING OF 10 ROTOR DIAMETERS

Parameter	Turbine 1	Turbine 2	Turbine 3	Turbine 4	Turbine 5	Windfarm efficiency
OIF	$\frac{1}{3}$	$\frac{1}{3}$	.	.	.	96.24%
Yaw angle (degrees)	15.17	0	.	.	.	
OIF	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.	.	94.88%
Yaw angle (degrees)	15.88	15.17	0	.	.	
OIF	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.	94.17%
Yaw angle (degrees)	16.13	15.88	15.17	0	.	
OIF	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	93.73%
Yaw angle (degrees)	16.26	16.13	15.88	15.17	0	

TABLE IV
NUMERICAL SIMULATIONS FOR SPACING OF 15 ROTOR DIAMETERS

Parameter	Turbine 1	Turbine 2	Turbine 3	Turbine 4	Turbine 5	Windfarm efficiency
OIF	$\frac{1}{3}$	$\frac{1}{3}$	.	.	.	96.46%
Yaw angle (degrees)	14.22	0	.	.	.	
OIF	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.	.	95.10%
Yaw angle (degrees)	15.36	14.22	0	.	.	
OIF	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.	94.38%
Yaw angle (degrees)	15.78	15.36	14.22	0	.	
OIF	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	93.92%
Yaw angle (degrees)	15.99	15.78	15.36	14.22	0	

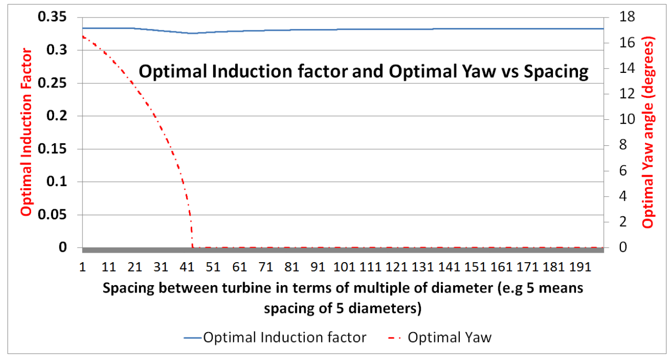


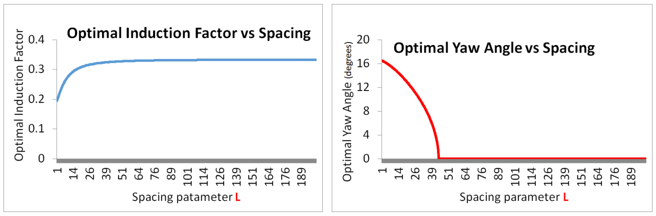
Fig. 4. Change in values of optimal induction factor and yaw angle with spacing for a windfarm with 2 turbines

important observation is that the optimal induction factor (OIF) is approximately equal to $\frac{1}{3}$ for all windfarms in both tables. This is also the value of OIF for a single turbine operated individually in free-stream wind without consideration of any wake effects. This shows that changing the yaw angle of a wind turbine deflects the wake in such a way that the downstream turbine is almost completely out of the wake region of the upstream turbine. This allows the upstream turbines to operate at the OIF of $\frac{1}{3}$. It is also because the increase in the power of the downstream turbine due to the rotation of upstream turbine offsets the decrease in the power of upstream turbine i.e, a net gain in total power is achieved.

The results show that rotating the wind turbine deflects the wake in such a way that the downstream turbine is almost completely out of the wake region of the upstream turbine. This allows the upstream turbines to operate at the optimal induction factor of $\frac{1}{3}$ because the downstream turbine experiences very little wake effect. We calculated the optimal induction factor and yaw angle for different values of spacing multiplier L for a two turbine windfarm. The plots are shown in Fig. 4.

It can be observed from Fig. 4 that the induction factor is approximately equal to a value of $\frac{1}{3}$, while yaw angle is the variable which changes its value to achieve maximum power with variation in spacing. Fig. 5a. shows the plot of optimal induction factor versus the spacing multiplier L for a two turbine case when the yaw angle is held constant at zero degrees. It can be seen that the optimal induction factor increases with spacing until it reaches the asymptotic value of $\frac{1}{3}$. The mathematical proof for the optimum induction factor when the spacing between turbines is very large is given in appendix A. Similarly Fig. 5b. shows the plot of optimal yaw angle pattern versus the spacing multiplier L for a two turbine case when the induction factor is held constant at $\frac{1}{3}$. It can be seen that the optimal yaw angle decreases with spacing until it reaches zero degrees. The mathematical proof for the optimum yaw angle when spacing between turbines is very large is given in appendix B.

Comparison of Fig. 5a. and Fig. 5b. with Fig. 4. shows that when the windfarm is jointly optimized using both yaw angle and induction factor, the yaw angle maintains its optimization trend in the same way as if windfarm was optimized using only



(a) (b)

Fig. 5. Change in values of optimal induction factor and yaw angle with spacing when optimized separately for a windfarm with 2 turbines

yaw angle as the optimization variable with induction factor set to a value of $\frac{1}{3}$. During the joint optimization in Fig. 4, the induction factor is approximately $\frac{1}{3}$ which is different from its optimum value in Fig. 5b. when yaw angle was held constant at zero degrees. Thus it is easy to conclude that yaw angle acts as the dominating optimization variable and the increase in power is achieved by changing only yaw angle and keeping the induction factor to a value of $\frac{1}{3}$.

This is an important result as it reduces the joint optimization of yaw angle and induction factor to that of only optimizing yaw angle. The values from Table II, III and IV and Fig. 4, 5a and 5b reveal that we can consider the yaw angle of wind turbines as the only control variable and set the induction factor equal to $\frac{1}{3}$. Setting $a_i = \frac{1}{3} \; \forall \; 1 \leq i \leq N$ in (3) – (5) gives us the following expression for wind velocity at turbine i in the windfarm in Fig. 3.

$$U_i = \begin{cases} U_{i+1}[1 - \frac{2}{3}\left(\frac{1}{1+2kL\cos\phi_{i+1}}\right)^2 & \\ \times \cos^2(4.5\,\phi_{i+1})] & \phi_{i+1} < 20^\circ, \\ U_{i+1} & \phi_{i+1} \geq 20^\circ, \end{cases} \tag{7}$$

where $\phi_{i+1} = 1.2\,\theta_{i+1}$ from (3).

From (5), the power generated by the i^{th} turbine will be

$$g_i = \frac{8}{27}\rho A \times U_i^3 \times \cos^2\theta_i. \tag{8}$$

IV. Dynamic Programming Formulation

We provide a Dynamic Programming Formulation to optimize the total power of a windfarm with N turbines using yaw angle as the optimization variable. Input variable in (8) is yaw angle θ_i and state variable is velocity U_i, given by (7). Using $\phi_{i+1} = 1.2\,\theta_{i+1}$, we have $\forall\; 0 \leq 1.2\,\theta_{i+1} < 20^\circ$

$$U_i = U_{i+1}(1 - \alpha_{i+1}\; \cos^2(5.4\,\theta_{i+1})), \tag{9}$$

where $\alpha_{i+1} = \dfrac{2}{3}\left(\dfrac{1}{1+2kL\;\cos(1.2\,\theta_{i+1})}\right)^2$.

Note that:

$$k > 0, L > 0 \Rightarrow 0 \leq \alpha_{i+1} < 1 \quad \forall\, 0 \leq 1.2\,\theta_{i+1} < 20^\circ$$

Velocity at turbine i, U_i is a function of upstream velocity, and the upstream input variable θ_{i+1}. The total power from turbine number 1 (WT_1) located at the end of the row to turbine number i (WT_i) is given as follows

$$G_i\;(\theta_1, \theta_2, \ldots \theta_i) = \sum_{m=1}^{i} g_m = \frac{8}{27}\rho A \sum_{m=1}^{i} U_m^3\;\cos^2\theta_m.$$

Define

$$h_i(\theta_1, \theta_2, \ldots \theta_i) = \frac{G_i(\theta_1, \theta_2, \ldots \theta_i)}{\frac{8}{27}\rho A U_i^3}. \tag{10}$$

U_i is independent of $(\theta_1, \theta_2, \ldots \theta_i)$ and $\dfrac{8}{27}\rho A$ is a constant term. Therefore optimizing $G_i(\theta_1, \theta_2, \ldots \theta_i)$ w.r.t. $(\theta_1, \theta_2, \ldots \theta_i)$ is equivalent to maximizing $h_i\;(\theta_1, \theta_2, \ldots \theta_i)$ w.r.t. $(\theta_1, \theta_2, \ldots \theta_i)$.

From definition of h_i, and (9),

$$h_i(\theta_1, \theta_2, \ldots \theta_i) = \cos^2\theta_i + f_i(\theta_i)\; h_{i-1}(\theta_1, \theta_2, \ldots \theta_{i-1}),$$

where

$$f_i(\theta_i) = \left(1 - \alpha_i\;\cos^2 5.4\,\theta_i\right)^3 \quad \forall\, i \geq 2.$$

For the wind turbine at the end of the row, $i = 1$ and

$$h_1 = \cos^2\theta_1$$
$$h_1^* = \max_{\theta_1}\{\cos^2\theta_1\}. \tag{11}$$

Let θ_1^* be the optimal value of θ_1 in (11). Then

$$\theta_1^* = 0, \quad \text{and} \quad h_1^* = \cos^2\theta_1^* = 1$$

We seek to establish the following dynamic programming (recursive optimization) relationship;

$$\max_{(\theta_1,\theta_2,\ldots\theta_{j+1})} h_{j+1}(\theta_1, \theta_2, \ldots \theta_{j+1}) = \max_{\theta_{j+1}}\{\cos^2\theta_{j+1} + f_{j+1}(\theta_{j+1}) \max_{(\theta_1,\theta_2,\ldots\theta_j)} h_j(\theta_1, \theta_2, \ldots \theta_j)\} \tag{12}$$

Let $(\theta_1^*, \theta_2^*, \ldots \theta_j^*)$ optimize $h_j(\theta_1, \theta_2, \ldots \theta_j)$ and c_1 be the optimum value. i.e,

$$c_1 = \max_{(\theta_1,\theta_2,\ldots\theta_j)} h_j(\theta_1, \theta_2, \ldots \theta_j)\} = h_j(\theta_1^*, \theta_2^*, \ldots \theta_j^*). \tag{13}$$

Let be θ_{j+1}^* chosen so as to maximize $\cos^2\theta_{j+1} + f_{j+1}(\theta_{j+1})\,c_1$. i.e,

$$\max_{\theta_{j+1}}\{\cos^2\theta_{j+1} + f_{j+1}(\theta_{j+1})\,c_1\} = \cos^2\theta_{j+1}^* + f_{j+1}(\theta_{j+1}^*)\,c_1. \tag{14}$$

CLAIM: We claim that $(\theta_1^*, \theta_2^*, \ldots \theta_{j+1}^*)$ from (13) and (14) optimize $h_j(\theta_1, \theta_2, \ldots \theta_{j+1})$ in (12) i.e,

$$\max_{(\theta_1,\theta_2,\ldots\theta_{j+1})} h_{j+1}(\theta_1, \theta_2, \ldots \theta_{j+1}) = h_{j+1}(\theta_1^*, \theta_2^*, \ldots \theta_{j+1}^*). \tag{15}$$

For the sake of contradiction, let there be $(\hat\theta_1, \hat\theta_2, \ldots \hat\theta_{j+1}) \neq (\theta_1^*, \theta_2^*, \ldots \theta_{j+1}^*)$. Such that

$$h_{j+1}(\hat\theta_1, \hat\theta_2, \ldots \hat\theta_{j+1}) > h_{j+1}(\theta_1^*, \theta_2^*, \ldots \theta_{j+1}^*). \tag{16}$$

Let

$$c_2 = h_j(\hat\theta_1, \hat\theta_2, \ldots \hat\theta_j) \tag{17}$$

From (14), we have

$$\cos^2\theta_{j+1}^* + f_{j+1}(\theta_{j+1}^*)\,c_1 > \cos^2\hat\theta_{j+1} + f_{j+1}(\hat\theta_{j+1})\,c_1. \tag{18}$$

From (13) $c_1 > c_2$. Therefore,

$$\cos^2\hat\theta_{j+1} + f_{j+1}(\hat\theta_{j+1})\,c_1 > \cos^2\hat\theta_{j+1} + f_{j+1}(\hat\theta_{j+1})\,c_2. \tag{19}$$

From (18) and (19)

$$h_{j+1}(\hat\theta_1, \hat\theta_2, \ldots \hat\theta_{j+1}) < h_{j+1}(\theta_1^*, \theta_2^*, \ldots \theta_{j+1}^*) \tag{20}$$

TABLE V
COMPARISON BETWEEN WINDFARM EFFICIENCY ACHIEVED FROM GREEDY CONTROL AND YAW ANGLE CONTROL USING DPF

Spacing Factor L	Number of Turbines N	Greedy control	Yaw control by using DPF
5	2	73.94%	96.07%
	3	56.94%	94.70%
	4	45.23%	94.00%
	5	37.41%	93.58%
10	2	85.65%	96.24%
	3	74.04%	94.88%
	4	64.59%	94.17%
	5	56.84%	93.73%
15	2	91.12%	96.46%
	3	83.29%	95.10%
	4	76.37%	94.38%
	5	70.24%	93.92%

This contradicts our assumptions from (16). Therefore, our assumption is incorrect and the claim in (15) is true. i.e,

$$\max_{(\theta_1,\theta_2,\dots\theta_{j+1})} h_{j+1}(\theta_1,\theta_2,\dots\theta_{j+1}) = h_{j+1}(\theta_1^*,\theta_2^*,\dots\theta_{j+1}^*)$$
$$= \cos^2\theta_{j+1}^* + f_{j+1}(\theta_{j+1}^*)\, h_j(\theta_1^*,\theta_2^*,\dots\theta_j^*)$$

From the above expression and (13)

$$\max_{(\theta_1,\theta_2,\dots\theta_{j+1})} h_{j+1}(\theta_1,\theta_2,\dots\theta_{j+1}) = \cos^2\theta_{j+1}^* + f_{j+1}(\theta_{j+1}^*) \max_{(\theta_1,\theta_2,\dots\theta_{j+1})} h_j(\theta_1,\theta_2,\dots\theta_j).$$

Using (14) in above expression gives

$$\max_{(\theta_1,\theta_2,\dots\theta_{j+1})} h_{j+1}(\theta_1,\theta_2,\dots\theta_{j+1}) = \max_{\theta_{j+1}}\{\cos^2\theta_{j+1} + f_{j+1}(\theta_{j+1}) \max_{(\theta_1,\theta_2,\dots\theta_j)} h_j(\theta_1,\theta_2,\dots\theta_j)\},$$

which is the same as (12).

V. COMPARISON OF RESULTS

In this section we will compare the results of our dynamic programming framework for optimization of total windfarm power through yaw angle control with the other techniques for optimizing power of windfarms. First, we compare the results of our formulation with the case in which every turbine is operated at an induction factor of $\frac{1}{3}$ and a yaw angle of zero degrees. This is the greedy control strategy [26] because every wind turbine is trying to maximize its individual power irrespective of the other turbines in the windfarm. These results are shown in Table V. Table V shows that there is a significant increase in the efficiency if the wind turbines operate according to the optimization framework provided in section IV. Moreover, the difference in windfarm efficiency increases as more wind turbines are added in the windfarm. This shows that for windfarms with a large number of turbines in a row, optimizing the total power by changing the yaw angle could result in a substantial increase in total output. For example, the total power in a windfarm with spacing of five rotor diameter is 1.3 times the total power from a greedy control strategy for a 2 turbine windfarm but it is 2.5 times more than the total power from a greedy control strategy for a 5 turbine windfarm.

TABLE VI
WINDFARM EFFICIENCY AND INDUCTION FACTOR FOR THE CASE WHEN ONLY INDUCTION FACTOR IS THE CONTROL VARIABLE

Spacing Factor L	Turbine 1	Turbine 2	Turbine 3	Turbine 4	Turbine 5	Windfarm Efficiency
5	0.243	$\frac{1}{3}$	.	.	.	76.77%
	0.193	0.243	$\frac{1}{3}$	.	.	62.45%
	0.160	0.193	0.243	$\frac{1}{3}$	.	52.68%
	0.137	0.160	0.193	0.243	$\frac{1}{3}$	45.59%
10	0.279	$\frac{1}{3}$	.	.	.	86.71%
	0.241	0.279	$\frac{1}{3}$	.	.	76.68%
	0.213	0.241	0.279	$\frac{1}{3}$	.	68.80%
	0.191	0.213	0.241	0.279	$\frac{1}{3}$	62.43%
15	0.298	$\frac{1}{3}$	.	.	.	91.57%
	0.270	0.298	$\frac{1}{3}$	.	.	81.53%
	0.247	0.270	0.298	$\frac{1}{3}$	.	78.54%
	0.229	0.247	0.270	0.298	$\frac{1}{3}$	73.39%

We also compare our proposed dynamic programming formulation with the optimization results of numerical simulations in which only induction factor acts as the optimization variable. First, we set the value of yaw angle to zero degrees in (4) and (5), which is the optimum value of yaw angle for a single turbine operating in free stream wind. This gives us

$$U_i = U_{i+1}\left(1 - 2a_{i+1}\left(\frac{1}{1+2kL}\right)^2\right) \tag{21}$$

$$g_i = \frac{1}{2}\rho A\left(4a_i\,(1-a_i)^2\right) \times U_i^3 \tag{22}$$

We vary the induction factor values from 0 to $\frac{1}{3}$ in the above equations for each wind turbine. We run numerical simulations to find the set of induction factor values for each wind turbine at which the total power of the windfarm is maximized for the four windfarms described in section III. We then calculate the windfarm efficiency using (6). These results are shown in Table VI. In Table VII we compare the results of our dynamic programming formulation for optimization using yaw angle with the case in which yaw angle is not chaWe also compare the total efficiency achieved from the proposed DPF with the case in which both yaw angle and induction factor act as control variables simultaneously.

We observe that for all three spacing values, optimizing only yaw angle to achieve maximum power using our dynamic programming framework gives the same values for maximum power if we allow both yaw angle and induction factor to change. This is because the optimum value of induction factor is approximately $\frac{1}{3}$. The results from Table VII also show that the windfarm efficiency achieved by only changing the induction factor and keeping the yaw angle of each turbine fixed at zero degrees is significantly less than the maximum efficiency calculated using our approach in section IV.

TABLE VII
COMPARISON OF DYNAMIC PROGRAMMING TECHNIQUE FOR YAW ANGLE OPTIMIZATION WITH THE OTHER CASES

Spacing Factor L	Number of Turbines N	Only Induction Factor	Only Yaw Angle	Both Induction Factor and Yaw Angle
5	2	76.77%	96.07%	96.07%
	3	62.45%	94.70%	94.70%
	4	52.68%	94.00%	94.00%
	5	45.59%	93.58%	93.58%
10	2	86.71%	96.24%	96.24%
	3	76.68%	94.88%	94.88%
	4	68.80%	94.17%	94.17%
	5	62.43%	93.73%	93.73%
15	2	91.57%	96.46%	96.46%
	3	81.53%	95.10%	95.10%
	4	78.54%	94.38%	94.38%
	5	73.39%	93.92%	93.92%

A. Benefits of spacing reduction.

We observe another important result from our research. It can be seen that the windfarm efficiency decreases significantly with the decrease in spacing between the turbines if the optimization is achieved by only changing the induction factor or if we use the greedy control strategy. However, there is not a significant decrease in windfarm efficiency due to reduced spacing when the total power is maximized using only yaw angle as the optimization variable.

For the dynamic programming optimization of yaw angle, the decrease in windfarm efficiency for all the windfarms in Table VII is not more than 0.4% when the spacing is reduced from 15 rotor diameters to 5 rotor diameters. From Table V, a maximum reduction of 32.83% in efficiency is observed for a 5 turbine windfarm when greedy control is employed. This means that if wind turbines are operated according to the yaw angle control technique of section IV, then they can be placed more closer without any significant loss in output. This would enable the windfarm planners to place the turbines more closely and increase the number of turbines in a windfarm during the design stage. Therefore, if the dynamic programming framework for yaw angle optimization is adopted, it would also result in efficient utilization of the windfarm area resources by allowing the spacing between the wind turbines to be reduced. This would be an additional advantage of our proposed scheme.

VI. DISCUSSION AND CONCLUSION

We included the effects of yaw angle in the commonly used Park and Jensen model [6] and obtained the velocity of a downstream turbine operating in the wake of an upstream turbine with rotated yaw. Our simulations show that the joint optimization of induction factor and yaw angle to maximize the total power of a windfarm results in an induction factor value approximately equal to $\frac{1}{3}$. In Fig. 4, we also showed that the induction factor maintains an approximate value of $\frac{1}{3}$ if the spacing between the wind turbines is changed but the yaw angle value changes with spacing to maximize the total power of the windfarm. Using this result, we reduced the problem of joint control of yaw angle and induction factor to a problem involving only yaw angle control. Our problem size was reduced from $2N$ variables to N variables and this allowed us to solve it using a dynamic programming technique to control the yaw angle of individual turbines. Using an induction factor value of $\frac{1}{3}$ for all the turbines, we proposed a dynamic programming framework to maximize the total power of a windfarm by optimizing the yaw angle of each turbine. In this way, we reduced the n variable problem into a n stage single variable problem.

Our results show that the windfarm efficiency by using the approach in section IV is approximately equal to the windfarm efficiency if both induction factor and yaw angle were optimized jointly. We also observed a large increase in windfarm efficiency compared to the efficiency values from greedy control strategy. The windfarm efficiency from the optimization scheme proposed in section IV is also higher than the technique in which the total power is optimized by only changing the induction factor and keeping the yaw angle constant at zero degrees. We also observe that the yaw angle optimization of individual turbines allows us to place the turbines closer to each other because the reduction in windfarm efficiency due to reduced spacing is not more than 0.4% for the windfarms considered in this paper. This is another important result for windfarm designing and planning and allows us to increase the number of wind turbines in a particular area without any significant loss in efficiency due to wake effects.

REFERENCES

[1] U. S. Department of Energy, 2012 Renewable Energy Data Book. [Online]. Available: http://www.nrel.gov/docs/fy14osti/60197.pdf

[2] U. S. Department of Energy, 20% wind energy by 2030: Increasing wind energy's contribution to U.S. electricity supply. [Online]. Available: http://www.nrel.gov/docs/fy08osti/41869.pdf

[3] T. Burton, D. Sharpe, N. Jenkins, and E. Bossanyi, *Wind Energy Handbook*. New York, NY, USA: Wiley, 2011.

[4] M. Steinbuch, W. de Boer, O. Bosgra, S. Peters, and J. Ploeg, "Optimal control of wind power plants," *J. Wind Eng. Ind. Aerodyn.*, vol. 27, no. 1-3, pp. 237–246, Jan. 1988.

[5] R. J. Barthelmie, K. Hansen, S. T. Frandsen, O. Rathmann, J. G. Schepers, W. Schlez, J. Phillips, K. Rados, A. Zervos, E. S. Politis, and P. K. Chaviaropoulos, "Modelling and measuring flow and wind turbine wakes in large wind farms offshore," *Wind Energy*, vol. 12, no. 5, pp. 431–444, 2009.

[6] N. O. Jensen, "A note on wind generator interactions," RISO National Labortory, Denmark, Tech. Rep. RISO-M-2411, Nov. 1983.

[7] J. W. Wagenaar, L. A. H. Machielse, and J. G. Schepers, "Controlling wind in ECN's scaled wind farm," in *Proc. EWEA*, Copenhagen, Denmark, Apr. 16–19, 2012.

[8] K. E. Johnson and N. Thomas, "Wind farm control: Addressing the aerodynamic interaction among wind turbines," in *Proc. Amer. Control Conf.*, St. Louis, MO, Jun. 10–12, 2009, pp. 2104–2109.

[9] D. Majidian and A. Rantzer, "A stationary turbine interaction model for control of wind farms," in *Proc. 18th IFAC World Congr.*, Milan, Italy, Aug. 2011.

[10] A. Kusiak, H. Y. Zheng, and Z. Song, "Power optimization of wind turbines with data mining and evolutionary computation," *Renew. Energy*, vol. 35, no. 3, pp. 695–702, 2010.

[11] P. O. Gebraad, F. Teeuwisse, J. van Wingerden, P. A. Fleming, S. D. Ruben, J. R. Marden, and L. Y. Pao, "A data-driven model for wind plant power optimization by yaw control," in *Proc. Amer. Control Conf.*, Portland, Oregon, Jun. 4–6, 2014, pp. 3128–3134.

[12] N. S. Moskalenko, K. Rudion, and Z. A. Styczynski, "Wind farm operation planning using optimal yaw angle pattern (OYAP)," in *IEEE PowerTech*, Trondheim, Norway, Jun. 19–23, 2011.

[13] D. Medici and J. Dahlberg, "Potential improvement of wind turbine array efficiency by active wake control," in *Proc. Eur. Wind Energy Conf.*, Madrid, Spain, Jun. 16–19, 2003.

[14] J. Park and K. H. Law, "A bayesian optimization approach for wind farm monitoring and power maximization," in *Proc. 6th Int. Conf. on Structural Health Monitoring of Intell. Infrastructure*, Hong Kong, China, Dec. 9-11, 2013.

[15] J. Park, S. Kwon, and K. H. Law, "Wind farm power maximization based on a cooperative static game approach," in *Proc. SPIE Active and Passive Smart Structures and Integr. Syst. Conf.*, San Diego, CA, Mar. 10–14, 2013.

[16] M. A. Rotea, "Dynamic programming framework for wind power maximization," in *Proc. 19th IFAC World Congr.*, Cape Town, South Africa, Aug. 24–29, 2014.

[17] E. Bitar and P. Seiler, "Coordinated control of a wind turbine array for power maximization," in *Proc. Amer. Control Conf.*, Washington, DC, Jun. 17–19, 2013, pp. 2898–2904.

[18] R. Bellman, *Dynamic Programming*. Dover, 2003.

[19] L. J. Vermeer, J. N. Sorensen, and A. Crespo, "Wind turbine wake aerodynamics," *Progress in Aerospace Sciences*, vol. 39, no. 6, pp. 467–510, 2003.

[20] WAsP wake effect model. [Online]. Available: http://www.wasp.dk/Software/WAsP/WakeEffectModel

[21] R. J. Barthelmie, S. C. Pryor, S. Frandsen, and G. C. Larsen, "Analytical modelling of large wind farm clusters," in *Proc. 2004 EAWE Conf., The science of making torque from wind*, Delft, Netherlands, Apr. 2004.

[22] H. P. Blomhoff, "An experimental investigation of wind turbine wakes," Master's thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2012.

[23] K. Boorsma, "Power and loads for wind turbines in yawed conditions," ECN, Petten, Netherlands, Tech. Rep. ECN-E–12-047, Nov. 2012.

[24] J. Choi and M. Shan, "Advancement of Jensen (Park) wake model," in *Proc. EWEA Annual Meeting*, Vienna, Austria, Feb. 4–7, 2013.

[25] M. Adaramola and P. Krogstad, "Experimental investigation of wake effects on wind turbine performance," *Renew. Energy*, vol. 36, no. 8, pp. 2078–2086, 2011.

[26] J. Marden, S. Ruben, and L. Pao, "A model-free approach to wind farm control using game theoretic methods," *IEEE Trans. Control Syst. Technol.*, vol. 21, no. 4, pp. 1207–1214, Jul. 2013.

APPENDIX A

Using (21) and (22) we can write the power equation for a two turbine windfarm with yaw angles set to zero degrees as.

$$G = \frac{1}{2}\rho A U_2^3 \left[4a_2(1-a_2)^2 + \frac{16}{27}\left\{1 - 2a_2\left(\frac{1}{1+2kL}\right)^2\right\}^3 \right]$$

For optimal a_2 : $\dfrac{\partial}{\partial a_2}\, G = 0$

$$4(1-a_2)(1-3a_2) + \frac{16}{9} \times \left[1 - 2a_2\left(\frac{1}{1+2kL}\right)^2\right]^2 \times 2\left(\frac{1}{1+2kL}\right)^2 = 0$$

Now as $L \to \infty$ $\left(\frac{1}{1+2kL}\right)^2 \to 0$. So

$$\lim_{L\to\infty} 4(1-a_2)(1-3a_2) + \frac{16}{9} \times [1 - 2a_2 \times 0]^2 \times 2 \times 0 = 0$$

And

$$a_1 = \frac{1}{3}$$

APPENDIX B

From (7), we have an expression for the velocity faced by the wind turbine when induction factor is held at $\frac{1}{3}$. Approximating $\cos\phi_{i+1} \approx 1$, we have

$$U_i = U_{i+1}\left[1 - \frac{2}{3}\left(\frac{1}{1+2kL}\right)^2\right] \times \cos^2(5.4\,\theta_{i+1})$$

Using (8), the power for a two turbine windfarm is

$$G = \frac{8}{27}\rho A U_2^3 \left[\cos^2\theta_2 + 1 - \frac{2}{3}\left(\frac{1}{1+2kL}\right)^2 \cos^2 5.4\theta_2\right]^3$$

For optimal yaw angle : $\dfrac{\partial}{\partial \theta_2}\, G = 0$

$$-\sin(2\theta_2) + 3\left[1 - \frac{2}{3}\left(\frac{1}{1+2kL}\right)^2 \cos^2 5.4\theta_2\right]^2 \times \left[\frac{2}{3} \times 5.4\left(\frac{1}{1+2kL}\right)^2 \sin(10.8\,\theta_2)\right] = 0$$

Or

$$\sin(2\theta_2) = 10.8\left(\frac{1}{1+2kL}\right)^2 \sin(10.8\,\theta_2) \times \left[1 - \frac{2}{3}\left(\frac{1}{1+2kL}\right)^2 \cos^2 5.4\theta_2\right]^2$$

Now as $L \to \infty$ $\left(\frac{1}{1+2kL}\right)^2 \to 0$. So

$$\lim_{L\to\infty} \sin(2\theta_2) = 10.8 \lim_{L\to\infty}\left(\frac{1}{1+2kL}\right)^2 \sin(10.8\,\theta_2) \times \lim_{L\to\infty}\left[1 - \frac{2}{3}\left(\frac{1}{1+2kL}\right)^2 \cos^2 5.4\theta_2\right]^2$$

This leads to

$$\lim_{L\to\infty} \sin(2\theta_2) = 0 \quad \Rightarrow \quad \theta_2 = 0$$