Journal Title: Visual Form Analysis and Recognition

Volume: 1
Issue:
Month/Year: 1992
Pages: 409-419

Article Author: George Nagy

Article Title: The Dimensions of Shape and Form

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Mukkai Krishnamoorthy (krishm)
Dept of Computer Science
RPI
Troy, NY 12180-3590

THE DIMENSIONS OF SHAPE AND FORM

George Nagy

Rensselaer Polytechnic Institute
Troy, NY, USA 12180

INTRODUCTION

Shape is used in computer vision to describe and identify objects. This paper is simply a collection of borrowed observations on the nature of shape. Many different points of view are discussed, but no definitive conclusions are reached. We request the reader's indulgence and begin with some definitions.

> SHAPE: *The outline or characteristic surface configuration of a thing: a contour; form.* (American Heritage Dictionary)

> SHAPE: *External form or contour; that quality of a material object (or geometric figure) which depends on constant relations of position and proportionate distance among all the points composing its outline or its external surface; a particular variety of this quality.* (Oxford English Dictionary)

In English, *shape* and *form* can be considered synonyms, but in Italian there is only the single word *forma*. For the purpose of this paper, we adopt a slightly more operational definition, as follows:

> SHAPE is a property of both *a set of objects* and *a particular method of observation or measurement.*

> The universe of objects may be finite (e.g., a set of printed characters on a page), or infinite (the set of all triangles). The objects are geometric figures or solids (compact point-sets in E^2 or E^3).

> The observations are restricted to the *geometry* of the envelope (boundary) of the objects. Only measurements that are invariant to translation, scale, and rotation are considered.

> Then all objects that cannot be differentiated by the given method of observation are said to have the same shape.

Thus we consider shape in terms of *equivalence classes* induced on a particular context or application (the universe of objects) by a particular system of features or measurements. Our definition does not reflect the idea that shape is a global property that is not affected by insignificant variations of the boundary. Indeed, the notion of *significant variation* is recognized to be thoroughly application-dependent. For instance, the short glyph that distinguishes the silhouette of a *Q* from an *O* has no effect when appended to an *R*.

> *The challenge is to find descriptions that are invariant to transformations leaving figures unaltered in "unimportant" ways, yet sensitive to transformations that change figures in "important" ways"* [Duda 73].

Our ultimate objective is a *quantitative* treatment of shape that will contribute to computer vision. Much of the discussion will be in terms of two dimensions. We have nevertheless resisted the temptation to include illustrations because they bypass analytical description.

Visual Form, Edited by C. Arcelli *et al*
Plenum Press, New York

In computer vision, image processing techniques are applied to 2-D transforms of 3-D objects. Projected shadow boundaries provide useful clues to shape. Digitized silhouettes of 3-D objects can be obtained from multiple vantage points [Shafer 1985]. Alternatively, 3-D surface coordinates can be obtained from reflectance maps where the third dimension of an object with uniform albedo is transformed into a grey-scale image. If the albedo is non-uniform, photometric stereo (shape from shading) with at least three separate lighting conditions is still workable [Horn 82]. With structured light, the object modulates a pattern projected upon it [Will 71]. The displacements of the pattern correspond to the third dimension.

The classical technique for measuring depth to obtain shape is *photogrammetric stereo* using one or more cameras. With a single camera a sequence of images is obtained from several known positions by mounting the camera on a robot arm or a coordinate measuring machine. If several cameras are used, their relative position is usually fixed. Although in image-formation systems the focus is a function of distance, obtaining *depth from blur* is more appropriate for textured and patterned surfaces than for surfaces with uniform albedo [Subbarao 88, Stewart 89, Nair 90].

Truly 3-D observation methods are also used in computer vision. *Scanning range-finders* use active probing by means of sound or light pulses [Jain 89]. *Diffraction-based methods* are also available [De Witt 88]. With these techniques, the distance of the surface is obtained directly, but reconstruction of a complete object still requires observations from several angles.

Often techniques are combined. For instance, silhouette information is insufficient to reconstruct depressions, so for complete characterization of complex objects it must be combined with other techniques. In model-based vision, the derived shape information is compared to the shape of objects stored in the form of *geometric models* (see below) to verify that the correct object is in the field of view (*verification*), to determine that the object is according to specification (*inspection*), to distinguish among several candidates (*identification*), to develop a symbolic description for higher-level processes (*scene understanding*), or to guide a tool or manipulator (*robotics*).

Our definition of shape rests on two pillars, the universe of objects and the method of observation. These are discussed in the next two sections, bearing in mind that:

> *...the notion that a numerical result should depend on the relation of the object to the observer is in the spirit of physics in this century and is even an exemplary illustration of it* [Mandelbrot 82].

OBJECT DOMAINS

It is clearly out of question to enumerate all the contexts in which the quantification of shape is of interest. The examples we draw on to illustrate the concept of object domains are the shapes of living organisms, of the earth (landforms), of crystals, and of the symbols of written communication.

Sir D'Arcy Wentworth Thompson devoted much of his long life to exploring the constraints that guide the development of *organic form*:

> *The study of form may be descriptive merely, or it may become analytical. We begin by describing the shape of an object in the simple words of common speech: we end by defining it in the precise language of mathematics; and the one method tends to follow the other in strict scientific order and historical continuity. Thus, for instance, the form of the earth, or a raindrop or a rainbow, the shape of the hanging chain, or the path of a stone thrown up into air, may all be described, however inadaquately, in common words; but when we have learned to comprehend and to define the sphere, the catenary, or the parabola, we have made a wonderful and perhaps a manifold advance. The mathematical definition of a "form" has a quality of precision which was quite lacking in our earlier stage of mere description; it is expressed in few words or in still briefer symbols, and these words or symbols are so pregnant with meaning that thought itself is economised; we are brought by means of it in touch with Galileo's aphorism (as old as Plato, as old as Pythagoras, as old perhaps as the wisdom of the Egyptians) that the "book of Nature is written in characters of Geometry."* [Thompson 42].

Applying the general principle of *least action*, Sir D'Arcy systematically investigates geometric similarity and scaling, compound interest, gravity, electrostatics, thermodynamics, hydrodynamics, and molecular forces. He draws attention to *the wondrous perfection of living organisms as an inevitable consequence of growth and evolution under these laws of nature*, in contrast to the popular perception that the "laws" of nature are descriptive,

not prescriptive. His *magnum opus* abounds in delightful insights about the cantilevered skeletons of camels, giraffes and kangaroos that optimize resistance to stress while avoiding shear; the variations in the shape of eggs due to the peristalsis of a uniform cylindrical oviduct; the resemblance of certain polyps to a freeze-time photograph of a splash of milk; the frugality of bees in setting the angles of the rhombic dodecahedra of their honey-combs; the helical spirals of phyllotaxis (the spiral tesselations of fir cones; "the curving rows of florets in the discoidal inflorescence of sunflowers"); and why an elephant is not built like a huge mouse.

According to Sir D'Arcy, Goethe was the first to call the study of organic form *morphology*. In contrast, spires, domes, promontories, escarpments, craters, buttes, mesas, and canyons are the subject of *physiography* (the study of landforms), and *geomorphology* (the role of geological evolution). The characterization of such shapes has stubbornly resisted quantification, yet for many purposes they describe landscape more informatively and concisely than the most accurate digital terrain elevation model or topographic map. Of course, verbal descriptions may be augmented with numerical information: the *height* of a peak, or the *direction* and *length* of a ridge. To a first approximation, *terrain* can be represented as a smooth scalar function of two variables. (At small scales, two spherical coordinates, latitude and longitude, are used, but at larger scales Cartesian axes on cartographic projections often suffice.) The third variable does not scale well: almost always higher resolution is required for elevation than for position, but the threshold between a mountain and a hill is region dependent.

The principal difficulty in the extraction of physiographic features is that we cannot consider only segmented, isolated objects. Ridges and valleys do not neatly tile the terrain: a chasm necessarily includes its walls; a crater, its rim. Sketch maps have long been a popular means of showing prominent features of the landscape, but it was only recently that attempts have been made to formalize this representation [Marr 76, Haralick 83]. (A graph-theoretic *primal sketch* for the gross shape of point sets is described in [Toussaint 88].)

Although there is nothing inherently periodic about most landforms, Fourier transforms with appropriate *window functions* allow approximate reconstruction. The ratios of the transform coefficients provide a rough indication of shape [Fredericksen 81]. However, considering the terrain as a continuous analytic function gives rise to an elegant mathematical representation in terms of critical points, course lines, and ridge lines. In the resulting segmentation into *slope districts*, region boundaries correspond to the arcs of the *critical point configuration graph* [Nackman 84]. Other types of *watershed models* are based on local classification according to first and second directional derivatives, or on *mathematical morphology* (see below) [Vincent 90].

Piecewise linear approximations divide the surface into triangular patches [De Floriani 89a]. In such a discrete approximation, each triangle can be specified using only a few parameters. Larger regions of homogeneous characteristics (flats, inclines, ridges and valleys) can then be assembled according to the *characteristic region configuration graph* [Falcidieno 90]. This method may be considered a discrete *dual* of the critical configuration graph mentioned above.

Much of the tumultuous irregularity of landscape is based on simple uniform building blocks such as the proverbial grains of sand. In nature, crystals such as quartz form prismatic polyhedra. Garnets are rhombic dodecahedra. *Crystallography* recognizes seven crystal systems (e.g., *triclinic*, *rhombohedral*) defined by *primitive lattices*. They are subdivided into 32 *point groups* and 230 *space groups* [Yale 68]. All essential distinctions can be visualized in terms of combinations of reflective and n-fold rotational symmetries of the lattice points (n takes on the values 1, 2, 3, 4, and 6, and, since very recently, for *quasi-crystals*, also 5.)

The experimental study of crystals hinges on the concentration of external or internal reflection and diffraction in preferred directions. Both visible and X-ray irradiation are used. The analysis of periodicities in X-ray crytallography was one of the first applications of the fast Fourier transform [Stout 68].

Let us now turn from natural objects to the shapes of the artificial. Precious stones are artfully cut and polished to maximize their brilliance from multiple directions. Popular shapes include *round*, *pear*, *marquis*, *teardrop*, *emerald* and *baguette* cuts. In high-quality gemstones, in contrast to rhinestones, the light is reflected not only from the surface, but also from multiple internal layers (cf. the *depth* of a diamond), but the external shape of a crystal is only tenuously related to its molecular-level symmetries.

Few developments have advanced civilization more than the invention of writing. The shapes used to record thought evolved in many directions in different cultures, and are evolving still [Knuth 79]. The development of machines that read has challenged inventors since at least the beginning of the nineteenth century.

Abstractly speaking, an *alphabet* consists simply of a fixed number of *differentiable* symbols. In writing and printing each member of the alphabet is rendered by one of an infinite set of images that share only the elusive quality of shape. In the Roman, Greek and Cyrillic alphabets there are only a few letters that are represented by markedly different shapes: one familiar example is A, a, a and a. In Arabic and, to a much lesser extent, in Devanagiri (a script derived from Sanskrit) the shape of a letter may depend on its neighbors according to a well-formed graphic grammar. (This also holds for *ligatures* in some Roman typfaces).

The study of invariant features that describe handprinted and particularly cursive characters remains a topic of continuing interest [Simon 89, Boccignone 90, Chianese 90]. However, almost any attempt at verbal or formal mathematical definition of the shape of any particular letter will occasionally fail: there will be images that obey the description that will be instantly recognizable as belonging to another class, and images that do not obey the description that would be corectly classified by any human. An instructive illustration showing how any of the ten numerals can be deformed into any of the others through a series of continuous transformations is found in [Wright 52].

A more subtle aspect of shape is the graphic unity of all of the alphabetic characters of a given typeface, such as *Palatino*, *Helvetica* or *Bodoni Bold*. Type designers strive to achieve such unity yet give their typeface a distinctive personality suitable for a particular context. Among the shape features used to distingish typefaces are the aspect-ratios of the characters, the lengths of the ascenders and descenders, the ratios of the widths of horizontal, vertical and slanted strokes (which, even in a given typeface, must be altered slightly from point-size to point-size to preserve the *illusion* of sameness), the size and form of serifs, the eccentricity of ovals, the curvature of hooks, and so forth [Scott 81].

Ideographs derive from pictorial representation of concrete objects, but in most Chinese characters the original inspiration is quite unrecognizable. Even in print, the strokes have a characteristic flow intended to evoke the trace of a brush dipped in ink. The symbols are distinguished and indexed by the number, orientation, and disposition of the strokes (but finding a character in the dictionary can be a formidable task).

In Chinese poems, the shapes and meanings of the individual ideographs combine with their sequence and overall disposition for the desired poetic effect. Calligraphy is considered the foundation of Chinese painting. Nowadays, calligraphy often appears in *distich*, two semantically symmetric sentences configured on a scroll.

A calligraphic piece must look lively and show both strength and fluidity. Saying that "the work penetrates the paper" is a compliment. To show strength, the hand must follow the writing brush, the wrist exerts substantial force, and the breath is synchronized with the rhythm of writing. To achieve fluency, the elbow (for large characters) or the wrist (for small characters) hangs losely. Hooks are traced quickly, while the brush halts for a pretty corner.

Examples of other fields that have given rise to specialized vocabularies of shape are *architecture* (the varieties of domes, columns, arches and fenestration); *pottery* (vases and urns that characterize archeological epochs); *neuro-anatomy* (the convoluted gyri and sulci of the primate brain); *phrenology* (the bumps of the human skull); *dactylography* (fingerprints); and study of *the multifarious and impermanent shapes of clouds*. In the next section, we describe some approaches that have been proposed for formalizing our uncanny ability to make consistent distinctions, based on shape, in these and other domains.

METHODS OF OBSERVATION

We begin with the classical tools of mathematics and proceed to more recent developments. In Euclidian Geometry, *similarity invariance* is a fundamental notion related to shape. It has its algebraic counterpart in the *group of similarity transformations*. Indeed, the different branches of geometry can be regarded as the study of those properties which are preserved under appropriate groups (*isometric*, *projective*, *affine*) of transformations [Klein 1878]. An example of a simple *metric* shape property in 2-D that is preserved under isometric transformations is the *aspect ratio*. It takes on the specific value celebrated as the *golden ratio* for a rectangle such that, when a square is cut off, the resulting rectangle is similar to the original. (The related sequence of integers are the equally celebrated Fibonacci numbers.) The golden ratio also surfaces in the dimensions of the regular polyhedra (see below).

Symmetries and *invariants* are fundamental aspects of shape that bridge geometry and group theory. In plane Euclidian geometry, these concepts may reach their acme in the study of *tilings* (a *plane tiling* is a countable family of closed sets which cover the plane without gaps or overlaps). The complete work on this topic is the

beautifully illustrated and heavily referenced [Grunbaum 87]. Twirling a kaleidoscope gives a glimpse of the inherent visual appeal of multiple symmetries (but: *Tyger! Tyger! burning bright / In the forests of the night, / What immortal hand or eye / Dare frame thy fearful symmetry?* [Blake 1794])

Rotational and reflective symmetries have been cataloged for all the regular polyhedra and their derivatives. (*Regular solids* have identical vertices and also identical faces that are regular polygons.) The *tetrahedron* is a face-vertex self-dual. The *octahedron* is the dual of the *cube*, and the *icosahedron* is the dual of the *dodecahedron*, so each pair exhibits the same symmetries. The five Platonic solids are the only regular *convex* solids, but there are four non-convex *Kepler-Poinsot star polyhedra* that are pairwise duals of each other. Also known since antiquity are the thirteen convex Archimedean *semi-regular* solids characterized by *identical* vertices and faces composed of *several* types of regular polygons. Two of these have left-handed and right-handed versions due to their lack of reflective symmetries. Each Archimedean solid also has a dual with the same symmetries: for instance, the dual of the *great rhombicosidodecahedron* is the *hexakis icosahedron*. The duals of these isogonal solids exhibit a useful property that makes them suitable for *dice* with up to 120 outcomes: all their faces are identical and equidistant from the center.

Other families of solids can be derived from the above by *truncation* (beveling corners), *faceting* (bridging vertices with planes), and *stellation* (intersecting extrapolated face-planes). Cauchy showed that the Kepler-Poinsot solids can be obtained from the Platonic solids by stellation. Infinite families of semi-regular *prisms* and *antiprisms* were studied by Kepler. The duals of prisms are called *triangular dipyramids*, and the duals of anti-prisms are *trapezohedra*. These can be adapted (by combining a pair of faces for each outcome, *a la* Dungeons and Dragons) for making fair dice for an arbitrary number of choices. Superb models of all these solids (and of many more), can be found in [Holden 71]. Another fine pictorial introduction is [Pearce 78]. For *the* authoritative treatment of regular polytopes, see [Coxeter 63]. But whether your taste runs to images or formulas, don't miss Senechal's and Fleck's incomparable "polyglot and polydiscliplinary polyhedral anthology" [Senechal 88]!

The application of symmetries and group theory to planning robotic assembly has been developed in a series of lively papers [Popplestone 84, 88, 90, Liu 90]. Symmetry groups are also closely related to the lattice structures used in molecular chemistry and in crystallography. Perspectivities can be considered the basic symmetries of a projective space [Yale 68]. In the discussion of fractals below, we will see another invariant related to symmetry, *self-similarity*, that can be expressed in terms of the *renormalization group*.

Topological properties that remain invariant under rubber-sheet transformations are formally called (in 2-D) *homeomorphisms* of the plane onto itself. Important topological invariants are the number of connected components and the number of holes. It can be shown that no *local* operator can count these, but that their difference, the *Euler Number*, can be computed from local properties only. Formulas that relate the number of vertices, edges, faces and loops in two and three dimensions are called *Euler-Poincare Formulas* (*Euler-Schlaefli Formulas* in higher dimensions) and play an important role in solid modeling. Of course, simple polygons and polyhedra can also be classified according to the cardinality of various topological elements. (The Euler constraints may also be regarded as *statistical symmetry*: when the valencies between different topological elements all assume their average value, the result is "ordinary" symmetry [Loeb 76].) However, attempts to describe the shape of commonplace objects, like letters of the alphabet, in only topological terms, without any *metric* observations, have been generally unsuccessful.

Certain shape-related properties of 2-D algebraic curves can be extracted from the coefficients of their equations in parametric form. Degeneracies and singular points such as nodes and cusps can be predicted. For non-singular planar curves, there is a shape taxonomy of classes (*topological type* or *form*) of curves that are homeomorphic under a homeomorphism that entends to the whole plane. These cataloging methods have also been extended to curves with singularities consisting of topological double points [Gonzales-Corbalan 90]. Some parametric curves whose names derive from their shapes are *rose curves*, *limacons* (including the special case of the *cardioid*), *cycloids*, *lemniscates*, and *spirals of Archimedes*.

Space curves and surfaces are, of course, more complex, but quadratic forms are commonplace. Plato showed that there are only six surfaces of revolution (i.e., surfaces symmetrical about an axis) engendered by *roulottes* traced by the foci of conic sections rolling along an axis: the *plane*, the *sphere*, the *cylinder*, the *catenoid*, the *unduloid*, and the *nodoid*. These are just the shapes assumed by a blown-glass bottle, a soap-bubble between wire rings, and many small unicellular organisms.

Differential Geometry explores the relationships of normal and tangent vectors of arbitrary surfaces. Surface points where the maximum and minimum (principal) curvatures are equal are called *umbilics*. *Ruled* and

developable surface approximations are used in shaping objects by multiple passes of a cutting tool. *Splines, Ferguson curves, Bezier curves*, and piecewise surface approximations derived from them guarantee the various continuity properties that characterize *sculptured surfaces* [Faux 79, Farin 90]. A thorough but concise mathematical treatment of the underlying group theory and tensor calculus is [Guggenheimer 66].

Fractal Geometry was invented by Benoit Mandelbrot to characterize a family of shapes. Fractal shapes can be generated by either deterministic or pseudo-random algorithms: in the latter case, both their regularities and irregularities have a statistical basis. Some of the roots of fractal geometry can be traced to the physics of Brownian motion and turbulence. Although many of the underlying ideas have been gathering dust on mathematicians' shelves for centuries (Cantor sets, Weierstrass functions, Peano curves, Koch islands, Hausdorff-Besikovitch dimensions), Mandelbrot demonstrated their application to natural phenomena from landscapes to snow flakes [Mandelbrot 82]. He argued that *pathological* shapes - pimply, grainy, pocky, tangled, wispy, wrinkled shapes - occur more commonly in nature than do Euclid's prissy figures and solids.

A key idea is that of *fractal dimension* which, unlike topological dimension, may take on fractional values. It is related to the exponent of a quantity epsilon (the resolution of the measuring tool) that will give a consistent estimate of length (or area or volume) as epsilon approaches zero. The fractal dimension always exceeds the topological dimension. For a simple space-filling curve, for example, it is 2.0 while the topological dimension is 1.

Another important notion is that of *self-similarity*, the property of a fractal invariant under ordinary geometric similarity. The equivalence classes defined by self-similarity go further than ordinary notions of symmetry: for the celebrated example of natural coast lines (How Long is the Coast of Britain? [Mandelbrot 67]), it can be argued that the *scale* at which a segment of coast line was represented cannot be determined by any set of observations on that segment. (The fractal dimension of typical coast lines is about 1.3.)

Computational Geometry offers some insight on the shape of complex Euclidian objects by taking advantage of properties that are not readily calculated by hand. Efficient algorithms and data structures have been developed for the convex hull, triangulations of point sets and polygons, Voronoi diagrams, and planar visibility [O'Rourke 85, Preparata 85]. The difference between a figure and its convex hull is called the *convex deficiency* [Calabi 68]. Each connected component of the convex deficiency may, in turn, be described recursively by its convex hull and deficiency. Polygons can also be decomposed into simpler *perceptually meaningful* components by analysis of the relative neighborhood graph of its vertices [ElGindy 88]. Many other decomposition methods for polygonal objects are discussed in [Keil 85].

Specialized taxonomies can be defined for dozens of different types of polygons (*star-shaped, monotone, crab, palm*), including hierarchies of polygonal shapes that can be triangulated in linear time [ElGindy 90]. The lengths of geodesic paths (shortest *internal* paths) have been suggested as an appropropriate measure of the shape of complex objects [Toussaint 86].

The shape of a set of points in the plane can be described by a generalization of the convex hull, based on generalized disks, called *alpha-hulls*. A related rectilinear construct, *alpha-shapes*, leads *to a spectrum of progressively more detailed description of the external shape of a given point set* [Toussaint 86]. Both have close counterparts in mathematical morphology.

Likewise, planar-faced 3-D objects can be characterized by decomposition into convex polyhedra. The decompositions may be space-based and regular (oct-trees [Samet 90]) or object-based and irregular (tetrahedra [Bruzzone 89, Eggenhofer 89]). These are envelope-based techniques, and therefore acceptable according to our definition. The boundary of sculptured objects is generally represented by polynomial or rational-polynomial *surface patches* [Farin 90]. For piecewise-linear triangulated surfaces, the notion of *terrain visiblity* can be used to define equivalence classes [Allen 86].

In Pattern Recognition the equivalence classes are usually formed according to a similarity or distance measure (such as the *Mahalanobis distance*) with respect to a set of *reference patterns*. Although a distance measure may be computed using an intermediate transformation (*features*), for blobs (2-D) and globs (3-D) the classification often attempts to capture the intuitive notion of shape. This is true for statistical methods, which attempt to deemphasize insignificant variations, as well as for structural and syntactic methods, where multiple alternative deterministic descriptions are formulated [Duda 73, Pavlidis 77]. In feature space, the statistical equivalent of geometric aspect ratios are the ratios of the magnitudes of the eigenvalues of covariance matrices. (These ratios are also related to the condition numbers of the matrices.) They provide the first approximation to the shape of multidimensional probability distributions, and are useful in dimensionality reduction and feature extraction.

The features used for the classification of 2-D and 3-D objects (as opposed to speech or radar signals) are often considered the domain of Image Processing [Pavlidis 80, 82]. Various boundary description methods, such as *chain codes*, lend themselves readily to descriptions in terms of consecutive convex and concave segments (lakes and bays), whose relative size is one characteristic of shape [Freeman 61, 74, 78]. Chains of line segments can be approximated to any degree of abstraction by forming chains of fewer segments subject to a constraint on the maximum deviation between the two chains [Toussaint 80].

Perhaps no single shape descriptor captures the imagination more than the Medial Axis Transform proposed in 1967 [Blum 67]. The medial axis (*MA*, or *skeleton*) can be defined as the locus of interior points equidistant from two or more points on the boundary of a figure [Patrikalakis 90]. It may be visualized as the *quench points* of a prairie fire started simultaneously all along the boundary [Duda 73]. This is equivalent to Blum's original definition of the medial axis transform (MAT) as the locus of the centers of all *maximal* disks that fit inside the boundary, and their associated *radial function* [Blum 67, 73]. In the *labeled skeleton*, the distance of the nearest point on the boundary (i.e., the radial function or quench function) is recorded with the curve itself, permitting complete reconstruction of the boundary. In 2-D, the medial axis consists of line segments, and in 3-D of surfaces. The medial axis transform is intimately related to Voronoi diagrams [Lee 82, Arcelli 86]. An analogous concept for point sets is the *minimal spanning tree* [Zahn 71, Toussaint 80]. Much effort has gone into the problem of the efficient (including parallel) computation of the medial axis for digitized images [Pfaltz 67, Arcelli 80, Samet 82, Arcelli 85], polygons [Montenari 69, Preparata 77, Kirpatrick 79, Bookstein 79], figures bounded by curves [Patrikalakis 90], and 3-D objects [Nackman 85, Hoffman 90].

> *The MAT captures the essence of shape. The MA and associated RF of a shape provide an idealized, yet complete, description of shape. The MA and the RF of a planar shape contain information on boundary curvature, its rate of change, flexure, and width properties of the shape. The MAT preserves the symmetry and periodicity of shape.* [Patrikalakis 90]

The only problem is that the medial axis may be no simpler to manipulate than the original boundary itself! However, *geometric* properties of the boundary are transformed into *topological* properties of the MAT. Decomposition of the figure may be simplified because different parts of an object correspond to different arcs of the MAT viewed as a graph (or, for simply-connected figures, as a tree) [Arcelli 82a]. The MAT may also be smoothed to eliminate minor variations in the boundary, and has been extensively used for shape representation [Calabi 68, Blum 78, Arcelli 80, Arcelli 82b, Heidi 84, Pizer 87]. Another natural application is thinning digitized patterns for optical character recognition, a problem far more complex (both to state precisely and to solve) than might appear at first [Arcelli 81].

Standard integral transforms for approximating functions can, of course, be applied to boundary curves. It is convenient to consider closed boundaries as periodic functions of some angular measure that repeats on multiple traces around the curve. The boundary of a blob is therefore often characterized in terms of its *radial distance* from a well-defined point (such as the centroid) as a function of angle. For curvilinear objects, Moment or Fourier expansions are often used. According to our criterion, blobs can be considered to have the same shape if their Fourier expansions up to some limiting frequency [Persoon 77, Kryzak 99, Reeves 88], or moment expansion up to some order [Hu 62, Teague 79], are identical. For multiple invariances, either multiple expansions, i.e., transforms of transforms, or appropriately weighted combinations of terms, are necessary.

Efficient methods based on integral transforms have been developed for size, rotation and translation invariant comparisons of digitized images [Khotanzad 90], of polygonal shapes [Arkin 90]. These ideas can be extended, of course, to 3-D, but have only recently found application in tomography. An alternative approach is based on morphological shape decomposition [Pitas 90].

Mathematical Morphology may be regarded as a set-theoretic formalization of filtering operations on multidimensional objects represented as arrays with binary or multivalued entries [Serra 82, Sternberg 86]. The operations of interest can generally be implemented by translation of a "structuring element" consisting of a (usually small) template over the object array. One may view the structuring element, in the spirit of our definition of shape, as the instrument for observation, or *probe* [Sinha 90]. Morphological *dilation* (sometimes called *dilatation*) with a disk-shaped structuring element tends to enlarge the object isotropically, while erosion tends to shrink it. Minor boundary variations tend to disappear in the process, while the overall object characteristics can be maintained by alternating the application of the two operators. Consequently shape can be defined as those characteristics that are left invariant under such transformations.

Appropriately used, mathematical morphological operations tend to simplify image data preserving their essential shape characteristics and eliminating irrelevancies [Haralick 87].

Dilation in mathematical morphology is closely related to integral *measures* such as the Minkowski Sum. Integral Geometry, however, concerns the probability distributions of geometric events, as in the celebrated computation of the transcendal number pi in the Bouffon Needle Problem [Matheron 75]. It is not far-fetched to argue that the distribution of the lengths of the intersections of a set of randomly-placed line segments with a geometric figure depends on the *shape* of that figure [Novikoff 62, Duda 73]. Indeed, for some shapes such distributions can be analytically predicted, and the placing of random lines stopped once the shape is known with sufficient certainty [Wong 69]. Furthermore, the observed distributions will be affected only insignificantly by small imperfections of the boundary: in other words, the method is robust.

In Solid Modeling a number of algorithms have been developed to extract *form features* [Wozny 90]. Some of the algorithms depend exclusively on the *adjacency topology* of the objects described in terms of the relationship of vertex, edge, and face primitives [Mantyla 88]. Typical features include *protrusions* and *depressions*, *slots*, *through-holes*, and *handles* [De Floriani 89b, Falcidieno 89]. One application is automating the planning of machining (material removal and forming) operations.

Computer Graphics applications of shape consist primarily of representation by means of abstraction. *Icons* are used in desk-top metaphors, and increasingly in other user-interface programs, instead of verbal descriptions [Chang 89, Glinert 90]. *Stick figures* represent complex sequences of movement in choreography, and in the study of some motion disabilities. *Computer-animated cartoons* interpolate changes in the shape of characters due to movement. The drawing of *caricatures*, while not yet automated, fall in the category of shape abstraction. It is also safe to say that without computer-graphics tools, mathematical morphology and fractal geometry would have taken much longer to achieve their current level of popularity.

Finally, art for art's sake is gradually becoming a legitimate field of computer graphics. See, for instance, the annual ACM Siggraph exhibits, the Catalog of the 1983 touring exhibition *On Art and Design* (J. Lipsky, project director) and the extraordinary volume of studies in shape by William Latham [Latham 89]. One cannot help wondering whether computers would have been the medium of choice for the likes of Albrecht Durer and M.C. Escher!

In spite of all our attempts to formalize it, one may argue that shape is essentially a *perceptual* phenomenon. Extensive psycho-physical, neuro-physiological and psychological experimentation over the last 150 years have shed some light on the cognitive aspects of shape [Helmholtz 1867, Julesz 71, Levine 85]. Physiological mechanisms have been formulated to explain perspective invariance. Among the most significant findings is that some shape-related features are detected quite early in the visual system, and that retinal images undergo a series of successive mappings in layers of the visual cortex [Hubel 79].

Gestalt phenomena (the whole is greater than the sum of its parts) have been investigated using figure-background experiments and visual illusions. Rorschach ink-blot tests are said to give insight on a person's thought processes A series of ingenious experiments with computer-generated dot patterns have revealed pscyho-physical relation of stereo vision to the perception of texture [Julesz 62].

Researchers have also attempted to predict perceived shape (*perceptual similarity*) from ambiguous line drawings [Leuwenberg 71]. Such drawings can be described by devising a family of codes that represent the *regularity* and *hierarchy* of the patterns and postulating that the perceived shape is the *simplest* interpretation [Hatfield 85]. This requires, of course, a quantification of simplicity. The theory of *accessibility* attempts to match regularity and hierarchy in the code of the pattern to regularity and hierarchy in the pattern itself, using *iterations* (series of identical symbolic descriptors), *symmetries* (mirror-image sequences), and *alternations* (series containing subseries that start or end identically) [Van der Helm 86, 89].

CONCLUSION

There are, as promised, no conclusions. Methods that succeed in one domain may prove useless in another. The very notion of shape appears to be amorphous rather than cleanly limned. Each of us brings to its study our particular predilection. Yet there still lurks the underlying hope of learning the unifying principles that make shape the cornerstone of human visual perception. We are looking forward to discovering further dimensions to shape at this Workshop and to forming and reforming our views on this provocative topic.

ACKNOWLEDGMENTS

The author is indebted to Carlo Arcelli, Luigi Cordella, and Gabriella Sanniti di Baja for the digressive discussions over a decade that finally aroused his interest in shape. He is also grateful to Randolph Franklin, Nebi Gursoy, Dze-Mou Jung, Jill Nagy, Rebecca and Sharad Seth, Jean-Claude Simon, and Godfried Toussaint for suggestions and references that were of great help in writing this paper.

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