# Wavelet Decomposition and Reconstruction Using Arbitrary Kernels, a New Approach

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The discrete wavelet decomposition and reconstruction remains one of the main issues of current signal and image compression. A lot of work was done with the purpose of finding wavelet kernels that would guarantee good compression results. The question of orthogonal wavelet transform was treated in detail in [5], [4], [2] and other articles, while the biorthogonal wavelets were considered in [6], [7] and so on. Both approaches have the same drawback. They put too many restrictions on the choice of the wavelet filters. The orthogonality condition makes the filters to be either very long or having bad compression properties (regularity and the number of vanishing moments). While the biorthogonal filters mitigate this drawback to a certain degree, one still should go through a careful process of choosing them.

In this paper we present a method that allows to obtain reconstruction filter for virtually every decomposition filter. Also we consider several compression examples using the SPIHT compression algorithm (see [3]), some of which produce results that are better than those of Daubechies 7/9 filter (which is known to be one of the best among the existing filters).

The filter bank, or wavelet processing, usually involves filtering a signal with low and high pass analysis or decomposition filters, discarding every other sample on the output of each of them. The obtained filter coefficients might be processed somehow, e.g. for the purpose of compression or noise reduction, and then the original signal is supposed to be reconstructed from the coefficients by passing low and high band 2-interpolated coefficients through the correspondingly low and high pass synthesis (reconstruction) filters and summing the results.

For every discrete function $f$ we denote by a subscript $l$ its $l$-shift to the right: $f_l[k] = f[k-l]$. Also for any pair of discrete complex valued functions $f$

and $b$ we denote their inner product by

$$< f, b >= \sum_k f[k]\bar{b}[k]. \tag{1}$$

**Definition 1** *For any filter $f$, let us call the function*

$$A(f)[n] =< f, f_{2n} >, \tag{2}$$

*its even-shift autocorrelation function and its 2 sided Z-transform,*

$$A(f)(z) = \sum_{n=-\infty}^{\infty} A(f)[n]z^{-n}, \tag{3}$$

*its even-shift moment generating function.*

The following theorem states that using any pair of mirror filters for decomposition and reconstruction, and inserting some recursive filtering in the middle, we can obtain perfect reconstruction.

**Theorem 1** *Taking $g$ and its mirror filter $h[k] = (-1)^k g[1-k]$ to be the decomposition filter bank and $G(z)$ and $H(z)$, their Z-transforms. We obtain perfect reconstruction, by taking the reconstruction low and high pass filter $g^{(r)}$ and $h^{(r)}$ With the following Z-transforms:*

$$G^{(r)}(z) = \frac{G(z)}{A(g)(z)} \tag{4}$$

$$H^{(r)}(z) = \frac{H(z)}{A(g)(z)}. \tag{5}$$

The proof of the theorem is based on the theory developed in [8] and is beyond the scope of this summary.

The theorem gives a simple algorithm to perform decomposition and reconstruction: we can take $g^{(d)}$ to be any low pass filter. We take $h^{(d)}$ to be its mirror filter. Then we do the decomposition (filtering and 2-decimation). Then, we apply the filter $b$ whose Z-transform is given by $B(z) = 1/A(g)(z)$ factor in 4 and 5, 2-interpolate the coefficients in both subbands, pass them through the same filters $g^{(d)}$ and $h^{(d)}$ and sum up the results. Filter $b$ can be implemented as a recursive bi-directional (zero-phase) filter.

One can see that the recursive filtering ($\frac{1}{A(g)(z)}$ factor in the 4 and 5) can be performed also on the decomposition stage. We noticed that the best compression results are obtained when the recursive filtering is done on the decomposition stage for the low bands and on the reconstruction stage for the high bands.

By running the algorithm with different filters and supports up to 7, we found filters that produce the best results by trial and error. Below we have a table

| *filter comparison results* | |
|---|---|
| Filter | PEAK-TO-PEAK SNR (dB) |
| Daubechies 9/7 | 33.69 |
| [1 2 1] | 33.09 |
| [-1 2 10 10 2 -1] | 33.59 |
| [-1 -0.5 6 11 6 -0.5 -1] | 33.78 |

of SPIHT compression results for Lena image with the rate of 0.25 bits/pixel. Our filters are denoted by the tap values of the unnormalized low-pass filters of the mirror pair. The normalization serves the purpose of making the sum of the taps equal $\sqrt{2}$. The recursive filtering is performed on the decomposition stage for the low bands and on the reconstruction stage for high bands. The results were obtained without entropy coding.

# References


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