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Analysis of Linear Prediction, Coding and Spectral Estimation from Subbands *

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Abstract

Linear Prediction schemes make a prediction $\hat{x}_i$ of a data sample x_i using p previous samples. It has been shown [1, 3] that as the order of prediction $p \to \infty$, there is no gain to be obtained by coding subband samples. This paper deals with the less well understood theory of finite-order prediction and optimal coding from subbands which are generated by ideal (brickwall) filtering of a stationary Gaussian source. We first prove that p^{th} order prediction from subbands is superior to p^{th} order prediction in the fullband, when p is finite. This fact adduces that optimal vector p-tuple coding in the subbands is shown to offer quantifiable gains over optimal fullband p-tuple coding, again when p is finite. The properties of subband spectra are analysed using the spectral flatness measure. These results are used to prove that Subband DPCM provides a coding gain over Fullband DPCM, for finite orders of prediction. In addition, the proofs provide means of quantifying the subband advantages in linear prediction, optimal coding, and DPCM coding in the form of gain formulas. Subband decomposition of a source is shown to result in a whitening of the composite subband spectrum. This implies that, for any stationary source, a p^{th} order Prediction Error Filter (PEF) can be found that is better than the p^{th} order PEF obtained by solving the Yule-Walker equations resulting from the fullband data. We demonstrate the existence of such a "super-optimal" PEF and provide algorithmic approaches to obtaining this PEF. The equivalence of linear prediction and AR spectral estimation is then exploited to show theoretically, and with simulations, that AR spectral estimation from subbands offers a gain over fullband AR spectral estimation.

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Index Terms

1. Subbands
2. Spectral Flatness Measure
3. Entropy Reduction
4. Finite order prediction
5. DPCM Coding Gain
6. Spectral Estimation

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1 Introduction

Subband decomposition is a technique in which the source spectrum is bandpass filtered and subsampled to obtain a time-frequency decomposition. Although the technique introduces delay into the system, this drawback is offset by the coding gains obtained from processing subband data rather than the fullband data. These observed gains have led to extensive use of subband decomposition for compression of video, image, audio and speech data [4, 5, 14]. However, there are only a few theoretical results [6, 14] in the literature to justify the widespread use of subbands.

Many subband schemes use Differential Pulse Code Modulation (DPCM) to code the subband samples. DPCM is a scheme where the difference between a data sample and linear prediction of this sample from past sample values is encoded. It has been shown [1, 3] that as the order of prediction, (i.e., the number of past samples available for the linear predictor), $p \to \infty$, there is no gain to be obtained by coding subband samples. However, in practice, considerable gains have been obtained from using DPCM in subbands over DPCM in the fullband, when the order of prediction is finite. These observations have so far been unsupported by theoretical analysis. This lack of theoretical justification for subband DPCM coding gain motivates our study of finite-order prediction in subbands.

In this paper, an information-theoretic framework is provided for the analysis of linear prediction and coding in subbands. The subband decomposition considered in this paper is a generalized scheme in which the source is ideally bandpass filtered into M subbands, not necessarily of equal frequency support. It is shown that the prediction error variance of the fullband always exceeds the total prediction error variance of the combined subbands, for a given prediction order p. This result is instrumental in proving that p^{th}-order entropy of the combined subbands is closer to the entropy rate of the source than the p^{th}-order entropy of the fullband, for any finite p, where p is the size of the block of source samples. This entropy reduction indicates that there is a quantifiable decrease in the rate for optimal vector p-tuple coding of subband samples versus the same coding of the fullband samples. This is equivalent to reduction of "memory" in the subbands and indicates a decorrelation of the source due to the subband decomposition. The spectral flatness measure (sfm) is used to prove properties of the subband spectra which result in the memory reduction in the subbands. These results are used to provide the first theoretical proof of the gain of subband DPCM over fullband DPCM for finite orders of prediction. The information-theoretic formulation also provides formulas for the computation of the coding gain of subband DPCM.

Because the subbands are decimated, the constraint of p-th order coding or prediction in each subband makes the number of such operations the same as compared to the same p-th order operation in the original fullband source. Of course, the subband analysis and synthesis filter banks imply additional complexity. We then consider the case where the sum of the orders of the subband operations equals the order of the fullband operation. It is a well known result that the solution of the Yule-Walker equations yields the optimal p^{th} order Prediction Error Filter (PEF) for the source, for a given prediction order p. We show that there exists a "super-optimal"

PEF obtained by combining the M optimal subband PEFs, the m^{th} subband PEF being of order p_m, with $\sum_{m=1}^{M} p_m = p$. This "super-optimal" PEF can perform better in terms of prediction error as well as computational complexity. However, the "super-optimal" PEF is unrealizable in practice since it calls for non-integer prediction orders and assumes the availability of fractionally indexed time samples. This motivates algorithmic approaches to the construction of the "super-optimal" filter, which are constrained to use integer prediction orders and available subband data samples. The problem of determining the optimal prediction order p_m for the m^{th} subband is formulated and two algorithms are proposed to solve this problem. The analogy between linear prediction and AR spectral analysis is exploited to prove the superiority of spectral estimation from subbands, both when the model order in the subbands and the fullband is p, and when $\sum_{m=1}^{M} p_m = p$. Simulations are provided to verify the theoretical results.

The paper is organized as follows. The next section is background material on subband decomposition and entropies. Some definitions and notational practices are introduced. In Section 3, the problem of p^{th} order entropy reduction is formulated in information-theoretic terms. Section 4 deals with subband linear prediction and PEFs. Here we present some inequalities that will help prove the p^{th} order result. The entropy-reduction proof is presented in Section 5, along with other results, which lead to the proof of subband DPCM coding gain. The existence of the "super-optimal" PEF is proved and discussed in Section 6. We propose two algorithms to optimally allocate prediction order within subbands under certain constraints. Section 7 presents the results of simulations with a high-order AR source that demonstrate the superiority of spectral estimation from subbands.

2 Background

In this section we briefly review subband decomposition and introduce our notation for the same. We also define p^{th} order entropies and the spectral flatness measure (sfm).

2.1 Subband Decomposition

The Power Spectral Density (psd) of the 1-D stationary Gaussian source $\mathbf{X}$ is denoted by $P_{xx}(\omega)$. The subbands are obtained by filtering the source spectrum into contiguous, non-overlapping frequency bands by means of a bank of ideal bandpass filters. Consider a bank of M filters with positive lower edge frequencies $\omega_1, \omega_2, \ldots, \omega_M$; $\omega_1 = 0$ and $\omega_{M+1} = \pi$. (Fig. 1). The m^{th} filter has the characteristic

$$H_m(\omega) = \begin{cases} 1, & \text{if } \omega \in I_m \\ 0, & \text{otherwise} \end{cases}$$

where I_m denotes the frequency range of the m^{th} subband and is given by

$$I_m = [\omega_m, \omega_{m+1}) \bigcup (-\omega_{m+1}, -\omega_m], \; m = 1, 2, \ldots, M. \tag{1}$$

Since the bandwidth of the m^{th} subband is now only $W_m = \omega_{m+1} - \omega_m$, we can subsample by a (rational) factor $V_m = \frac{\pi}{W_m}$ to obtain a fullband sequence over $-\pi$ to π referenced to the lower sampling frequency. This combination of filtering and subsampling is referred to as decimation (Fig. 2). The I_m satisfy

$$\begin{aligned} &\bigcup_{m=1}^{M} I_m = [-\pi, \pi] \quad \text{and} \\ &I_m I_n = \phi \quad \text{for} \quad m \neq n \end{aligned} \tag{2}$$

Filtering with the m^{th} ideal subband filter results in a Gaussian random sequence with psd

$$P_{xx}^{m}(\omega) = \begin{cases} P_{xx}(\omega), & \omega \in I_m \\ 0, & \omega \notin I_m \end{cases} \tag{3}$$

After resampling by V_m, we obtain a random subsequence with psd

$$P_{xx}^{(m)}(\omega) = \frac{1}{V_m} P_{xx}\left(\frac{\omega}{V_m} + \omega_m \mathrm{sgn}(\omega)\right). \tag{4}$$

The original input sequence can be reconstructed exactly by interpolating V_m zeros for the m^{th} subband sequence and re-filtering (Fig. 3). The gain of the m^{th} (ideal) reconstruction filter is made equal to the subsampling factor V_m for that subband. This filter gain reverses the $\frac{1}{V_m}$ scaling of the subband spectra, which is needed to preserve the variances.

2.2 p^{th} Order Entropies

For Gaussian sources, the p^{th} order per letter differential entropy of a block of p source samples is given by

$$h_p(\mathbf{X}) = \frac{1}{2} \ln 2\pi e |\Phi_p|^{\frac{1}{p}}, \tag{5}$$

where $|\Phi_p|$ is the p^{th} order covariance matrix of the source. For $p = 1$, we get the so-called first-order entropy and for $p \to \infty$, we arrive at the entropy rate of the source. The entropy rate of the Gaussian source is defined as

$$h_\infty(\mathbf{X}) = \frac{1}{2} \ln 2\pi e Q_X \tag{6}$$

where Q_X is the source entropy power and equals $\exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{xx}(\omega) d\omega\}$. Similarly, for the subbands, the p^{th} order per letter differential entropy for the m^{th} subband is

$$h_p^{(m)}(\mathbf{SB}) = \frac{1}{2} \ln 2\pi e |\Phi_p^{(m)}|^{\frac{1}{p}}. \tag{7}$$

The entropy power of the m^{th} subband is given by $Q_{\mathbf{SB}} = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{xx}^{(m)}(\omega) d\omega\}$ and the entropy rate of the m^{th} subband is then given by

$$h_{\infty}^{(m)}(\mathbf{SB}) = \frac{1}{2}\ln 2\pi e Q_{SB}^{(m)} \tag{8}$$

Because the subbands have each been subsampled by a factor of V_m, the contributions of their per-letter entropies to the combined per-letter entropy must each be reduced by this factor. Therefore the combined per-letter p^{th} order subband entropy is given by

$$h_p(\mathbf{SB}) = \sum_{m=1}^{M} \frac{1}{V_m} h_p^{(m)}(\mathbf{SB}), \tag{9}$$

for all non-negative integers p.

The spectral flatness measure (sfm) ξ_X of the source is defined as the following ratio

$$\xi_X = \frac{\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{xx}(\omega) d\omega\right]}{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega) d\omega} = \frac{Q_X}{\sigma_X^2}. \tag{10}$$

The sfm is a real number between 0 and 1. When the source spectrum is white, the sfm is 1 and when the source spectrum is maximally correlated, the sfm is 0. For the sfm to be 0, the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{xx}(\omega) d\omega$ must be $-\infty$. From the Paley-Wiener criterion, this means that the source is perfectly predictable. The inverse of the sfm is a measure of the predictability of the source. The distortion-rate function for the Gaussian source $\mathbf{X}$ is equal to $D(R) = 2^{-2R}\xi_X \sigma_X^2$, for mean squared error $D(R)$ and $D(R) \leq \min P_{xx}(\omega)$. Thus, smaller values of the sfm mean lesser entropy, more predictability and a smaller theoretical mean-square distortion D for a given rate R bits/sample.

The difference between the entropy rate and the p^{th} order entropy is the amount of information per letter that the rest of the sequence contains about the the first p samples [2]. i.e.,

$$h_p(\mathbf{X}) - h_{\infty}(\mathbf{X}) = \lim_{N\to\infty}\left[\frac{1}{p} I(X_1, X_2, \ldots, X_p; X_{p+1}, \ldots, X_{p+N})\right] \tag{11}$$

This difference, which we call the p^{th}-order memory, overbounds the rate difference between coding p-tuples of the source and optimally coding the source [2] and, for small distortions, equals this rate difference [3, 20]. In terms of the rate-distortion function, the p^{th}-order memory of the source can be written as

$$h_p(\mathbf{X}) - h_{\infty}(\mathbf{X}) = R_p^X(D) - R_{\infty}^X(D) \tag{12}$$

where $R_p^X(D)$ is the p^{th}-order rate distortion function for the source $\mathbf{X}$. The p^{th}-order memory of the combined subbands can similarly be expressed in terms of the rate-distortion function as

$$h_p(\mathbf{SB}) - h_{\infty}(\mathbf{SB}) = R_p^{SB}(D) - R_{\infty}^{SB}(D) \tag{13}$$

If this rate difference is smaller for subbands than for a fullband source, then there is a rate advantage in coding subband p-tuples as opposed to coding fullband p-tuples.

3 Framework of the problem

The problem confronting us can be stated as follows: How does a linear predictor, using p past values for prediction, perform compared to M linear predictors, each using p past samples of the subband sequences? In the scalar case, $p = 0$, the performance gain of subband PCM over fullband PCM can be shown to be a consequence of the fact that the combined first-order entropies of the subbands approach the source entropy [3, 6]. This approach can be generalized to the p^{th} order prediction problem and we propose to show that p^{th} order prediction from subbands is superior, which then allows us to deduce that the combined p^{th} order entropies of the subbands approach the entropy rate of the source more closely than the p^{th} order fullband entropy for any finite p.

When p-dimensional blocks of the source are available, the p^{th} order per letter differential entropy is given by

$$h_p(\mathbf{X}) = \frac{1}{2} \ln 2\pi e |\Phi_p|^{\frac{1}{p}} \tag{14}$$

where $|\Phi_p|$ is the p^{th} order covariance matrix of the source. Therefore, the difference between the per sample p^{th} order entropy and the entropy rate of the source is

$$\begin{aligned} {}^p\Delta_\infty^{\mathbf{X}} &= h_p(\mathbf{X}) - h_\infty(\mathbf{X}) \\ &= \frac{1}{2} \ln \frac{|\Phi_p|^{\frac{1}{p}}}{Q_{\mathbf{X}}} \end{aligned} \tag{15}$$

Similarly, for the m^{th} subband,

$${}^p\Delta_\infty^{(m)} = \frac{1}{2} \ln \frac{|\Phi_p^{(m)}|^{\frac{1}{p}}}{Q_{\mathbf{SB}}^{(m)}} \tag{16}$$

where $|\Phi_p^{(m)}|$ is the p^{th} order covariance matrix of the m^{th} subband source.

The rate difference per sample between the composite p^{th} order entropies of the subbands and the entropy rate of the source is [3]

$$\begin{aligned} {}^p\Delta_\infty^{SB} &= \sum_{m=1}^{M} \frac{1}{V_m} {}^p\Delta_\infty^{(m)} \\ &= \sum_{m=1}^{M} \frac{1}{2V_m} \ln \frac{|\Phi_p^{(m)}|^{\frac{1}{p}}}{Q_{\mathbf{SB}}^{(m)}} \end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned} {}^p\Delta_\infty^{U} - {}^p\Delta_\infty^{SB} &= \frac{1}{2} \ln \frac{|\Phi_p|^{\frac{1}{p}}}{\prod_{m=1}^{M} (V_m |\Phi_p^{(m)}|^{\frac{1}{p}})^{\frac{1}{V_m}}} \\ &= \Delta\mathcal{H} \end{aligned} \tag{18}$$

$\Delta\mathcal{H}$ is the quantity we define as the reduction in memory due to subbanding. If $\Delta\mathcal{H} \geq 0$, there is a rate advantage in going to the subbands for vector p-tuple coding. In Section 4.2, we shall present an important inequality that will help prove this fact. This inequality states that linear prediction from subbands is superior to linear prediction from the fullband.

4 Subband Linear Prediction

The linear prediction problem is that of finding the optimal set of coefficients $\{a[1], a[2], \ldots a[p]\}$ such that

$$\hat{x}(n) = -\sum_{i=1}^{p} a[i]x(n-i)$$

minimizes $E[|x(n)-\hat{x}(n)|^2]$. By using the fact that the error vector $e(n) = x(n)-\hat{x}(n)$ is orthogonal to the subspace spanned by $x(n-1), x(n-2), \ldots, x(n-p)$, we arrive at the Yule-Walker equations

$$\sum_{i=1}^{p} a(i)r_{xx}(k-i) = -r_{xx}(k), \;\; k = 1, 2, \ldots p. \tag{19}$$

The minimum prediction error power is given by

$$\rho_{min} = r_{xx}(0) + \sum_{i=1}^{p} a(i)r_{xx}(-i) \tag{20}$$

Here $r_{xx}(i)$ is the i^{th} correlation of the source and $r_{xx}(0)$ is equal to the source variance σ_x^2.

We can use the Levinson-Durbin algorithm to find the optimal predictor coefficients for all orders of prediction. The recursion yields the following set of predictor coefficients $\{a_1[1]\}, \{a_2[1], a_2[2]\}, \ldots, \{a_p[1], a_p[2], \ldots, a_p[p]\}$ where $a_k[i]$ is the i^{th} coefficient of the k^{th} order linear predictor. The prediction error power for the k^{th} order model is denoted by ρ_k, i.e., ρ_k is the minimum mean-squared-error over all linear estimators $\hat{x}(n)$ given $\{x(n-1), x(n-2), \ldots x(n-k)\}$, namely,

$$\rho_k \;\; = \;\; \mathrm{E}[(x(n)-\hat{x}(n))^2 / x(n-1), \ldots, x(n-k)]$$

The set $\{a_p[i]\}$ of optimal p^{th} order predictor coefficients results in an optimal p^{th} order Prediction Error Filter (PEF),

$$\tilde{A}(\omega) = 1 + \sum_{i=1}^{p} a_p[i] \exp(-j\omega i). \tag{21}$$

The optimal PEFs are be denoted by a "tilde" over the filter, i.e., $\tilde{A}(\omega)$.

If the source is, in fact, an AR(p) process, then the optimal p^{th} order PEF $\tilde{A}(\omega)$ will have coefficients equal to the AR parameters of the source process and the prediction error will be white with power ρ_p equal to the driving white noise variance σ^2. However, if the source is not an AR(p) process, then the prediction error will not be white. The prediction error spectrum will then be given by

$$P_{ee}(\omega) = |\tilde{A}(\omega)|^2 P_{xx}(\omega) \tag{22}$$

where $\tilde{A}(\omega) = 1 + \sum_{i=1}^{p} a(i)e^{-j\omega i}$ is the optimal p^{th} order PEF and $P_{xx}(\omega)$ is the source spectrum. The minimum prediction error power is equal to the variance of the prediction error spectrum obtained by using the optimal PEF $\tilde{A}(\omega)$ and is given by

$$\rho_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{ee}(\omega)d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde{A}(\omega)|^2 P_{xx}(\omega)d(\omega) \tag{23}$$

The prediction error can be regarded as a stochastic process of the same spectral support as $x(n)$ as seen by its power spectral density in (22) and with variance given in (23) as the integral over its power spectral density.

The autocorrelation matrix of the source has some connection to the prediction error variances. As before, the autocorrelation matrix of a block of p source samples is

$$\Phi_p = \begin{bmatrix} r_{xx}(0) & r_{xx}(1) & \dots & r_{xx}(p-1) \\ r_{xx}(1) & r_{xx}(0) & \dots & r_{xx}(p-2) \\ \vdots & \vdots & \vdots & \vdots \\ r_{xx}(p-1) & r_{xx}(p-1) & \dots & r_{xx}(0) \end{bmatrix} \tag{24}$$

The matrix A whose rows are solutions to the Yule-Walker equations of orders $p-1, p-2, \dots, 1, 0$ diagonalizes Φ_p , viz.,[10]

$$\Phi_p = (A^{-1})^T \Lambda A^{-1} \tag{25}$$

where

$$A = \begin{bmatrix} 1 & a_1[1] & a_2[2] & \dots & a_{p-1}[p-1] \\ 0 & 1 & a_2[1] & \dots & a_{p-2}[p-2] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{26}$$

and

$$\Lambda = diag(\rho_0, \rho_1, \dots, \rho_{p-1}) \tag{27}$$

where $\rho_i, i = 0, \dots, p-1$ are the corresponding error variances for prediction of order i. Note that $\rho_0 = r_{xx}(0)$. Since the determinant of A is unity, it follows that

$$|\Phi_p| = \prod_{i=0}^{p-1} \rho_i \tag{28}$$

The determinant of the autocorrelation matrix of order p is therefore the product of the prediction error variances of orders 0 through $p-1$. This determinant is also the product of the eigenvalues of Φ_p, but this fact does not prove as useful in the current context.

4.1 Subband Prediction

Let the source process now be filtered into subbands by the filter bank of ideal band-bass filters $H^m(\omega), m = 1, 2, \ldots, M$. We first establish a relation between the auto-correlations of the subbands *prior* to downsampling and those of the fullband source.

The k^{th} correlation of the source, $r_{xx}(k)$, is given by

$$\begin{aligned} r_{xx}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)e^{j\omega k}d\omega \\ &= \frac{1}{2\pi}\sum_{m=1}^{M}\int_{I_m} P_{xx}(\omega)e^{j\omega k}d\omega \end{aligned} \tag{29}$$

At the output of the filter, the k^{th} correlation is given by

$$\begin{aligned} r_{xx}^m(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H^m(\omega)|^2 P_{xx}(\omega)e^{j\omega k}d\omega \\ &= \frac{1}{2\pi}\int_{I_m} P_{xx}(\omega)e^{j\omega k}d\omega. \end{aligned} \tag{30}$$

Therefore,

$$r_{xx}(k) = \sum_{m=1}^{M} r_{xx}^m(k) \tag{31}$$

and

$$\Phi_p = \sum_{m=1}^{M} \Phi_p^m \tag{32}$$

where Φ_p and Φ_p^m are the $p \times p$ autocorrelation matrices of the source and bandpass filtered source, respectively.

Consider now a particular subband process $x^m(n)$ *before* subsampling and its p-th order prediction error process $e^m(n)$. We shall prove that the subsampled error process $e^{(m)}(n)$ for that subband has the same variance as $e^m(n)$, the error process before subsampling. [1]

For p^{th} order prediction in a subband, the prediction error spectrum before subsampling is

$$P_{ee}^m(\omega) = |\tilde{A}(\omega)|^2 P_{xx}^m(\omega) \tag{33}$$

and its variance is

$$\rho_p^m = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{ee}^m(\omega)d\omega \tag{34}$$

[1] We use the following notational convention:

- Bracketed superscripts $^{()}$ denote the subsampled subband domain and
- Unbracketed superscripts refer to processes and filters *without* subsampling.

The exceptions to these conventions can be understood from the context.

Following (4), the prediction error spectrum after subsampling by V_m is

$$P_{ee}^{(m)}(\omega) = \frac{1}{V_m} P_{ee}^{m}\left(\frac{\omega}{V_m} + \omega_m \mathrm{sgn}(\omega)\right). \tag{35}$$

Integration over $P_{ee}^{(m)}(\omega)$ yields the prediction error after subsampling of

$$\begin{aligned}\rho_p^{(m)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{ee}^{(m)}(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{V_m} P_{ee}^{m}\left(\frac{\omega}{V_m} + \omega_m sgn(\omega)\right)d\omega \\ &= \frac{1}{2\pi}\int_{I_m} P_{ee}^{m}(\omega)d\omega \\ &= \rho_p^m \end{aligned} \tag{36}$$

Therefore, the prediction error variances before and after subsampling are equal, as claimed. One should notice that the subsampled error process is not a linear combination of the subsampled subband elements $x^{(m)}(n)$, but is comprised of elements which are linear combinations of subband elements $x^m(n)$, some of which are no longer present in $x^{(m)}(n)$ after subsampling. However, it is evident that since these latter elements can be reconstructed with zero mean-squared error from linear combinations of subband elements, there exists a (possibly non-causal) linear filtering of the subband error process which will produce the prediction error spectrum as in (22) and hence the same variance. When one first forms the prediction from the decimated subband samples, the variance of the error also remains the same after upsampling and ideal bandpass filtering with gain factor V_m in reconstruction of the bandpass subband process.

Let us examine what happens when an AR process is decomposed into (ideal) subbands. Suppose the source is an AR(q) process, i.e.,

$$x_n = -\sum_{k=1}^{q} x_{n-k} a_k + e_n \tag{37}$$

where $\{a_k\}$ are the AR parameters of the source and $\{e_n\}$ is a white noise process with variance σ_e^2. Therefore, the source has a PSD

$$P_{xx}(\omega) = \frac{\sigma_e^2}{|A(\omega)|^2} \tag{38}$$

where $A(\omega) = 1 + \sum_{k=1}^{q} a_k \exp(-j\omega k)$.

After filtering by the m^{th} ideal bandpass filter $H^m(\omega)$, and then subsampling by V_m, the corresponding subsampling factor, we obtain the subband PSD as

$$P^{(m)}(\omega) = \frac{\sigma_e^2 |H^m\left(\frac{\omega}{V_m} + \omega_m \mathrm{sgn}(\omega)\right)|^2}{|A\left(\frac{\omega}{V_m} + \omega_m \mathrm{sgn}(\omega)\right)|^2}. \tag{39}$$

But $H^m\left(\frac{\omega}{V_m}+\omega_m \mathrm{sgn}(\omega)\right)$ is exactly equal to 1 over $-\pi<\omega\leq\pi$. Therefore,

$$P^{(m)}(\omega)=\frac{\sigma_e^2}{|A\left(\frac{\omega}{V_m}+\omega_m \mathrm{sgn}(\omega)\right)|^2}=\frac{\sigma_e^2}{|A^{(m)}(\omega)|^2}\quad -\pi<\omega\leq\pi \tag{40}$$

where

$$\begin{aligned} A^{(m)}(\omega) &= 1+\sum_{k=1}^{q} a_k \exp\left(-j[\frac{\omega}{V_m}+\omega_m \mathrm{sgn}(\omega)]k\right) \\ &= 1+\sum_{k=1}^{q}[a_k \exp(-j\omega_m \mathrm{sgn}(\omega)k)]\exp(-j\frac{\omega}{V_m}k) \end{aligned} \tag{41}$$

Since $A^{(m)}(\omega)$ is of maximum order $\frac{q}{V_m}$ referred to the fullband frequency axis, the PSD of the m^{th} subband is that of an all-pole process of *maximum* order $\frac{q}{V_m}$. It is not an AR process in the ordinary sense of (37), but only in the white noise filtering sense of (38). $A^{(m)}(\omega)$ is a polynomial of integer powers of $e^{j(\omega/V_m)}$ up to a maximum order of q. In effect, the order of the filter or denominator polynomial is $\frac{q}{V_m}$, and this filter is driven by white noise of variance σ_e^2. Note that the number of parameters required to define the so-called "subband all-pole process of maximum order $\frac{q}{V_m}$" is q. The above equations describe an AR process over a generalized subband decomposition. A similar definition for AR processes over dyadic trees (pyramid decomposition) can be found in [17, 18].

4.2 Prediction Error Inequality

Now, we proceed to prove the main result of this section.

Theorem 1 *For an AR(q) process, the minimum p^{th}-order prediction error of the fullband exceeds the aggregate of the minimum p^{th}-order prediction errors of the subbands, for $p\leq q$. i.e.,*

$$\rho_p \geq \sum_{m=1}^{M}\rho_p^{(m)} \tag{42}$$

where ρ_p is the minimum prediction error power of the fullband and $\rho_p^{(m)}$ is the minimum prediction error power of the m^{th} decimated subband.

Proof: [15]

Let $\mathcal{F}=\{^{\sqcap}\mathcal{A}(\omega)\}$, $u\in\mathcal{U}$ be the set of all possible p^{th} order PEFs. ($\mathcal{U}$ is an uncountably infinite index set). Let us denote the prediction error power obtained by using an arbitrary PEF ${}^vA(\omega)\in F$ by ${}^vr_{ee}(0)$. Consider the minimization of

$${}^vr_{ee}(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}|{}^vA(\omega)|^2 P_{xx}(\omega)d\omega \tag{43}$$

over the set $\mathcal{F}$. As shown previously, an AR(q) fullband source is still an all-pole process of order at most q in the (ideal) decimated subbands. Since any continuous source can be represented as an AR process [9] of possibly infinite order, this minimization is valid for all continuous sources.

Since we use ρ_p to denote the minimum prediction error power obtained by using the optimal p^{th} order PEF, we can write

$$\begin{aligned}
\rho_p &= \min_{v\in\mathcal{U}} {}^v r_{ee}(0) \\
&= \min_{v\in\mathcal{U}} \frac{1}{2\pi}\int_{-\pi}^{\pi} |{}^v A(\omega)|^2 P_{xx}(\omega)d\omega \\
&= \min_{v\in\mathcal{U}} \frac{1}{2\pi}\sum_{m=1}^{M}\int_{\omega\in I_m} |{}^v A(\omega)|^2 P_{xx}(\omega)d\omega \\
&= \min_{v\in\mathcal{U}} \frac{1}{2\pi}\sum_{m=1}^{M}\frac{1}{V_m}\int_{-\pi}^{\pi} |{}^v A(\frac{\omega}{V_m}+\omega_m \mathrm{sgn}(\omega))|^2 P_{xx}(\frac{\omega}{V_m}+\omega_m \mathrm{sgn}(\omega))d\omega \quad (44)\\
&= \min_{v\in\mathcal{U}} \sum_{m=1}^{M}\frac{1}{2\pi}\int_{-\pi}^{\pi} |{}^v A^{(m)}(\omega)|^2 P_{xx}^{(m)}(\omega)d\omega \quad (45)
\end{aligned}$$

The PEFs, $\{{}^v A^{(m)}(\omega)\}$, with ${}^v A^{(m)}(\omega) = {}^v A(\frac{\omega}{V_m}+\omega_m \mathrm{sgn}(\omega))$, are subsampled versions of ${}^v A(\omega)$. This follows from the multirate identity that time-invariant filters can be moved across sampling rate changes without affecting the impulse response. As shown earlier, these PEFs are of maximum polynomial order p/V_m, but need p parameters for their specification. Consequently, the zeros of $A(\frac{\omega}{V_m}+\omega_m \mathrm{sgn}(\omega))$ in the z-plane are constrained to be the order V_m roots of the zeroes of the fullband PEF $A(\omega)$. Therefore, the class of filters corresponding to ${}^v A^{(m)}$ is included in $\mathcal{F}$, the set of all possible p^{th} order PEFs. Now, since each term in the above sum is positive, it is clear that any single optimal ${}^w\tilde{A}(\omega)$ cannot produce a minimum of ${}^v r_{ee}(0)$ which is equal to or less than that which would be obtained from a separate minimization of each integral in the sum. When we perform the individual minimizations of each term in the sum, we get a set of optimal subband PEFs, $\{\tilde{A}^{(m)}(\omega)\}$, each of which is of order p.

Therefore, we can write

$$\begin{aligned}
\rho_p &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde{A}(\omega)|^2 P_{xx}(\omega)d\omega \\
&\geq \frac{1}{2\pi}\sum_{m=1}^{M}\int_{-\pi}^{\pi} |\tilde{A}^{(m)}(\omega)|^2 P_{xx}^{(m)}(\omega)d\omega \\
&= \sum_{m=1}^{M}\rho_p^{(m)} \quad (46)
\end{aligned}$$

where $\tilde{A}(\omega)$ is optimal for the fullband and the $\{\tilde{A}^{(m)}(\omega)\}$ are optimal for the subbands. $\rho_p^{(m)}$ is the minimum prediction error for the m^{th} subband, with $\{\tilde{A}^{(m)}(\omega)\}$ as its optimal PEF. Therefore,

$$\rho_p \geq \sum_{m=1}^{M}\rho_p^{(m)} \quad (47)$$

♠

The inequality derived above is instrumental in proving that $\Delta\mathcal{H}$, (defined in Section 3), is, in fact, always non-negative.

Corollary 1 *The above inequality is satisfied with equality if the source is white.*

Proof:

Let the source have a psd $P_{xx}(\omega) = \sigma_x^2$. When this source is subbanded, the resulting subbands are also white, with the m^{th} subband having a psd σ_m^2. For white spectra, higher order prediction offers no reduction in the prediction error variances. i.e.,

$$\begin{aligned}\rho_p &= \sigma_x^2 \\ \rho_p^{(m)} &= \sigma_m^2, \quad m = 1, \ldots, M\end{aligned}$$

The optimal PEF for a white source is one with a pole at zero. Therefore,

$$\begin{aligned}\rho_p &= \sigma_x^2 \\ &= \min_{k \in I} \frac{1}{2\pi} \sum_{m=1}^{M} \int_{-\pi}^{\pi} |^k A^{(m)}(\omega)|^2 \sigma_m^2 d\omega \\ &= \sum_{m=1}^{M} \sigma_m^2 \min_{k \in I} \frac{1}{2\pi} \int_{-\pi}^{\pi} |^k A^{(m)}(\omega)|^2 d\omega \\ &= \sum_{m=1}^{M} \sigma_m^2\end{aligned}$$

♠

5 Memory Reduction in Subbands

We begin this section with the proof of memory reduction in subbands, for Gaussian sources. We then investigate the spectral properties of subbands which help explain the entropy reduction.

Theorem 2 *For Gaussian sources, the composite p^{th} order entropies of the subbands are closer to the entropy rate of the source than the p^{th} order entropy of the fullband. i.e.,*

$$\Delta\mathcal{H} = ^p\Delta_\infty^X - ^p\Delta_\infty^{SB} \geq 0. \tag{48}$$

Proof:

Recall that $\Delta\mathcal{H}$ was given by $^p\Delta_\infty^X - ^p\Delta_\infty^{SB}$. Using $|\Phi_k| = \prod_{i=0}^{k-1} \rho_i$,we proceed as follows

$$\begin{aligned}^p\Delta_\infty^X - ^p\Delta_\infty^{SB} &= \frac{1}{2} \ln \frac{|\Phi_p|^{\frac{1}{p}}}{\prod_{m=1}^{M} (V_m |\Phi_p^{(m)}|^{\frac{1}{p}})^{\frac{1}{V_m}}} \\ &= \frac{1}{2} \ln \frac{\prod_{i=0}^{p-1} \rho_i^{\frac{1}{p}}}{\prod_{m=1}^{M} (V_m \prod_{i=0}^{p-1} (\rho_i^{(m)})^{\frac{1}{p}})^{\frac{1}{V_m}}}\end{aligned}$$

Since $\rho_i \geq \sum_{m=1}^{M} \rho_i^{(m)}$, we continue by writing

$$
\begin{aligned}
&\geq \frac{1}{2} \ln \frac{\prod_{i=0}^{p-1} \left(\sum_{m=1}^{M} \rho_i^{(m)}\right)^{\frac{1}{p}}}{\prod_{m=1}^{M} (V_m \prod_{i=0}^{p-1} (\rho_i^{(m)})^{\frac{1}{p}})^{\frac{1}{V_m}}} \\
&\geq \frac{1}{2} \ln \frac{\prod_{i=0}^{p-1} \left(\sum_{m=1}^{M} \rho_i^{(m)}\right)^{\frac{1}{p}}}{\sum_{m=1}^{M} \prod_{i=0}^{p-1} (\rho_i^{(m)})^{\frac{1}{p}}} \qquad (49)
\end{aligned}
$$

For the last inequality, we have used the Arithmetic-Geometric Mean inequality

$$
\sum_{m=1}^{M} \frac{K_m}{V_m} \geq \prod_{m=1}^{M} (K_m)^{\frac{1}{V_m}} \qquad (50)
$$

with $K_m = V_m (\prod_{i=0}^{p-1} \rho_i^{(m)})^{\frac{1}{p}}$, and where $\sum_{m=1}^{M} \frac{1}{V_m} = 1$. The argument of the logarithm in eqn.(49) is a ratio of product of sums to sums of products, where all of the terms are non-negative. When multiplied out, the numerator of the argument can be seen to include all of the terms of the denominator. Therefore $\Delta\mathcal{H} \geq 0$.

♠

The theorem implies that, by representing the source in the subband domain, we are reducing the source entropy. The nature of the memory reduction offered by subbanding can be stated in the form of the following corollaries.

Corollary 2 *Optimal vector p-tuple coding in the subbands offers a gain over optimal fullband vector p-tuple coding.*

Proof:

Rewriting $\Delta\mathcal{H}$ using eq.(12), we get

$$
\begin{aligned}
\Delta\mathcal{H} &= (h_p(X) - h_\infty(X)) - (h_p(SB) - h_\infty(SB)) \\
&= (R_p^X(D) - R_\infty^X(D)) - (R_p^{SB}(D) - R_\infty^{SB}(D)), \qquad (51)
\end{aligned}
$$

where $R_p^X(D)$ and $R_p^{SB}(D)$ are the p^{th}-order rate-distortion functions for the source (fullband) $\mathbf{X}$ and the combined subbands, respectively. Since $R_\infty^X(D) = R_\infty^{SB}(D)$ [20],

$$
\begin{aligned}
\Delta\mathcal{H} &= R_p^X(D) - R_p^{SB}(D) \\
&\geq 0 \qquad (52)
\end{aligned}
$$

Therefore there is a rate advantage in coding subband vector p-tuples versus vector p-tuple coding in the fullband and p is any nonegative finite integer (including 0).

Corollary 3 *Equality holds in the above theorem if the source spectrum is white.*

Proof:

$\Rightarrow$ When the source is i.i.d, the prediction errors are independent of the order of prediction and are all equal. i.e.,

$$
\begin{aligned}
\rho_i &= \sigma_x^2 \\
\rho_i^{(m)} &= \sigma_m^2 \qquad (53)
\end{aligned}
$$

for all $i = 1, \ldots, p$ and $m = 1, \ldots, M$. σ_x^2 is the source variance and σ_m^2 is the variance of the m^{th} subband. The subband variances are given by

$$\sigma_m^2 = \frac{\sigma_x^2}{V_m} \tag{54}$$

Therefore,

$$\begin{aligned}
\Delta\mathcal{H} &= \frac{1}{2}\ln\frac{|\Phi_p|^{\frac{1}{p}}}{\prod_{m=1}^{M}(V_m|\Phi_p^{(m)}|^{\frac{1}{p}})^{\frac{1}{V_m}}} \\
&= \frac{1}{2}\ln\frac{\sigma_x^2}{\prod_{m=1}^{M}(V_m\sigma_m^2)^{\frac{1}{V_m}}} \\
&= \frac{1}{2}\ln\frac{\sigma_x^2}{\prod_{m=1}^{M}(V_m\frac{\sigma_x^2}{V_m})^{\frac{1}{V_m}}} \\
&= \frac{1}{2}\ln\frac{\sigma_x^2}{\prod_{m=1}^{M}(\sigma_x^2)^{\frac{1}{V_m}}} \\
&= \frac{1}{2}\ln\frac{\sigma_x^2}{\sigma_x^2} \\
&= 0
\end{aligned} \tag{55}$$

♠

Corollary 4 $\Delta\mathcal{H}$ *asymptotically approaches 0 as $p \to \infty$.*

This asymptotic result is consistent with the result proved in [1].

Proof:

Consider eqn.(18). As $p \to \infty$, the determinants $|\Phi_p|^{\frac{1}{p}}$ and $|\Phi_p^{(m)}|^{\frac{1}{p}}$ approach the entropy power of the source and the entropy power of the m^{th} subband respectively [12]. The denominator of eqn.(18) is the geometric mean of the subband entropy pwers which equals the source entropy power. Therefore, asymptotically, as $p \to \infty$, $\Delta\mathcal{H}$ =0.

♠

Corollary 5 $\Delta\mathcal{H}$ *is monotonically increasing in M, the number of subbands, when the subbands are hierarchically ordered.*

Proof:

The M subbands can be defined as the partition of the frequency axis by $\{I_m, m = 1, 2, \ldots, M\}$. The $\{I_m, m = 1, 2, \ldots, M\}$ are non-overlapping intervals as defined earlier. Now consider a finer partition $\{I_n, n = 1, 2, \ldots, M+1\}$ obtained by further sub-dividing one of the $\{I_m\}$, say I_k. Let the two members of $\{I_n, n = 1, 2, \ldots, M+1\}$ obtained by splitting I_k be indexed as I_{s_1} and I_{s_2}. i.e.,

$$I_{s_1} \bigcup I_{s_2} = I_k \tag{56}$$

Clearly, this is equivalent to considering I_k as a fullband and I_{s_1}, I_{s_2} as subbands. From Theorem 2, therefore, there is an increase in the memory reduction. Therefore, $\Delta\mathcal{H}$ is monotonically increasing in M, the number of subbands.

♠

Remarks

- The condition that the new (finer) partition be obtainable by splitting members of the old partition is equivalent to saying that every new subband should be obtained by a filtering+subsampling operation on the old subbands, i.e., a hierarchical decomposition.
- The larger the number of subbands, the more gain there is to be obtained from subband prediction. The maximum possible gain is equal to the ratio of the arithmetic mean of the subband variances to their geometric mean.
- It is possible to find a partition of the frequency axis that is worse than a different one with fewer divisions.

The reason is that the best partition is one that maximizes the cumulative subband flatness measure. Increasing the number of subbands does increase the cumulative subband flatness measure, but maximizes it only in an asymptotic sense. For example, consider the spectrum in Fig. 4. The optimal subbands are obtained by ideal bandpass filters from $0 - \frac{\pi}{3}, \frac{\pi}{3} - \frac{2\pi}{3}$ and $\frac{2\pi}{3} - \pi$. A typical pyramid decomposition with equal subbands at each level will be optimal only asymptotically.

5.1 Spectral Properties of Subbands

The reason for memory reduction in subbands can be traced back to the "whitening" action of the subbanding process. In this section, we present some spectral properties of the subbands, in terms of the sfm. We shall need the following definitions : [15]

Definition 1 *The composite spectral flatness measure of the subbands, ξ_x^{SB}, is defined as the geometric mean of the individual sfm's of the subbands, $\xi_x^{(m)}$, $m = 1, \ldots, M$. i.e.,*

$$\xi_x^{SB} = \prod_{m=1}^{M} \left(\xi_x^{(m)}\right)^{\frac{1}{V_m}} \tag{57}$$

We introduce an alternate expression for the spectral flatness measure, for the Gaussian case, which is mathematically more convenient.

Lemma 1 *The spectral flatness measure ξ_x for a Gaussian source $\mathbf{X}$ can be expressed as*

$$\xi_x = \left(\frac{\exp[h_\infty(\mathbf{X})]}{\exp[h_1(\mathbf{X})]}\right)^2 \tag{58}$$

Remarks

Recall that the sfm is defined as the ratio of entropy power Q of the source and the source variance σ^2, viz.,

$$\xi = \frac{Q}{\sigma^2} \tag{59}$$

Substituting for the subband sfms in eqn.(57), we get

$$\xi^{SB} = \prod_{m=1}^{M} \left(\frac{Q_{SB}^{(m)}}{\sigma_m^2} \right)^{\frac{1}{V_m}} \tag{60}$$

where $Q_{SB}^{(m)}$ and σ_m^2 are, respectively, the entropy power and variance of the m^{th} subband. Therefore, the combined sfm ξ^{SB} is the ratio of the geometric mean of the subband entropy powers to the geometric mean of the subband variances. For Gaussian sources, this resolves to eqn.(58) in the second definition. i.e.,

$$\prod_{m=1}^{M} \left(\frac{Q_{SB}^{(m)}}{\sigma_m^2} \right)^{\frac{1}{V_m}} = \left(\frac{\exp[h_\infty^{SB}(\mathbf{X})]}{\exp[h_1^{SB}(\mathbf{X})]} \right)^2 \tag{61}$$

Theorem 3 *The spectral flatness measure (sfm) of a Gaussian source is a non-decreasing function of the subband process. i.e., if* $\xi_x^{(FB)}$ *denotes the sfm of the source, and* $\xi_x^{(SB)}$ *denotes the composite sfm of the subbands, then*

$$\xi_x^{(FB)} \leq \xi_x^{(SB)} \tag{62}$$

Proof:

Recall that the first-order source entropy, $h_1(\mathbf{X})$, is given by eqn.(5) with $p = 1$, and the entropy rate of the source, $h_\infty(\mathbf{X})$, is given by eqn.(6).

Therefore,

$$\frac{\exp[h_\infty(\mathbf{X})]}{\exp[h_1(\mathbf{X})]} = \left(\frac{Q_U}{\sigma^2} \right)^{\frac{1}{2}} \tag{63}$$

The RHS can be seen to be the square root of the sfm, ξ_x, of the source. From Theorem 2, with $p = 0$, the combined first-order entropy of the subbands is less than the first-order entropy of the source, i.e.,

$$h_1(\mathbf{SB}) \leq h_1(\mathbf{X}) \tag{64}$$

Here $h_1(\mathbf{SB})$ is the combined first-order entropy of the subbands. The first-order subband entropies combine according to eqn.(9).

Therefore,

$$\frac{\exp[h_\infty(\mathbf{X})]}{\exp[h_1(\mathbf{SB})]} = \sqrt{\xi_x^{(SB)}} \geq \frac{\exp[h_\infty(\mathbf{X})]}{\exp[h_1(\mathbf{X})]} = \sqrt{\xi_x^{(FB)}} \tag{65}$$

$\xi_x^{(SB)}$ is the composite spectral flatness measure of the subbands. It follows that $\xi_x^{(SB)} \geq \xi_x^{(FB)}$. In other words, the subbands are whiter than the fullband.

♠

Therefore, the spectrum of the combined subbands is flatter (whiter) than the fullband spectrum. Since the PEF's can be viewed as whitening filters, one would intuitively expect that, for a given order of prediction p, the following theorem to hold true.

Theorem 4 *The prediction error spectrum obtained by combining the individual subband prediction error spectra is flatter than the fullband prediction error spectrum. Equivalently, in terms of spectral flatness,*

$$\xi_e^{(SB)} \geq \xi_e^{(FB)} \tag{66}$$

where $\xi_e^{(SB)}$ is the sfm of the composite subband prediction error spectrum and $\xi_e^{(FB)}$ is the sfm of the fullband prediction error spectrum.

Proof:

Recall that the prediction error spectrum P_{ee} is given by

$$P_{ee}(\omega) = |A(\omega)|^2 P_{xx}(\omega) \tag{67}$$

where $A(\omega)$ is a PEF and $P_{xx}(\omega)$ is the spectrum of the time series input to the PEF. We make use of the following result [7]: If $A(\omega)$ is minimum phase, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln |A(\omega)|^2 d\omega = 0. \tag{68}$$

It can be shown [10] that the solution of the Yule-Walker equations results in a minimum phase filter. In other words, the prediction error power is always minimized by a minimum-phase filter. Therefore,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{ee}(\omega) d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\{|A(\omega)|^2 P_{xx}(\omega)\} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{xx}(\omega) d\omega \end{aligned}\tag{69}$$

After exponentiating and dividing by $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_{ee}(\omega)d\omega$, we obtain

$$\begin{aligned}\xi_e &= \frac{\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{xx}(\omega)d\omega\right]}{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_{ee}(\omega)d\omega} \\ &= \frac{\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{xx}(\omega)d\omega\right]}{\rho_p}\end{aligned}\tag{70}$$

where ρ_p is the prediction error variance for prediction order p. Similarly, the sfm of the error spectrum for the m^{th} subband is

$$\xi_e^{(m)} = \frac{\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{xx}^{(m)}(\omega)d\omega\right]}{\rho_p^{(m)}} \tag{71}$$

Since the composite sfm of the subbands is the geometric mean of the individual subband sfm's,

$$\begin{aligned}
\xi_e^{(SB)} &= \prod_{m=1}^{M} (\xi_e^{(m)})^{\frac{1}{V_m}} \\
&= \frac{\prod_{m=1}^{M} \exp\left[\frac{1}{2\pi V_m}\int_{-\pi}^{\pi} \ln P_{xx}^{(m)}(\omega)d\omega\right]}{\prod_{m=1}^{M}(\rho_p^{(m)})^{\frac{1}{V_m}}} \qquad (72)
\end{aligned}$$

The numerator resolves to $\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{xx}(\omega)d\omega\right]$. Therefore, the ratio of combined subband error sfm to fullband error sfm is given by

$$\begin{aligned}
\frac{\xi_e^{(SB)}}{\xi_e^{(FB)}} &= \frac{\rho_p}{\prod_{m=1}^{M}(\rho_p^{(m)})^{\frac{1}{V_m}}} \\
&\geq \frac{\sum_{m=1}^{M}\rho_p^{(m)}}{\prod_{m=1}^{M}(\rho_p^{(m)})^{\frac{1}{V_m}}} \\
&\geq \frac{\sum_{m=1}^{M}\rho_p^{(m)}}{\prod_{m=1}^{M}(V_m\rho_p^{(m)})^{\frac{1}{V_m}}} \\
&\geq 1 \qquad (73)
\end{aligned}$$

The first inequality follows from the prediction error theorem, while the last one is the Arithmetic Mean - Geometric Mean inequality. Therefore,

$$\xi_e^{(SB)} \geq \xi_e^{(FB)} \qquad (74)$$

♠

5.2 Subband DPCM v/s Fullband DPCM

At this point we are in a position to prove that DPCM coding of subbands for finite prediction orders is superior to DPCM coding of fullbands. In the previous theorem we showed that the combined subbands had an error spectrum whiter than the fullband error spectrum. Therefore, when the subband predictors are placed in a DPCM loop, the combined reconstruction error spectrum for the subbands is correspondingly whiter than the fullband reconstruction error spectrum.

Now, the scenario is similar to that of the PCM coding scheme, where we had M scalar quantizers for the subbands, each receiving a subband data stream. In the DPCM case, we have M subband DPCM loops, and the m^{th} subband scalar quantizer receives as its input the prediction error sequence for the m^{th} subband (Fig. 5). The fullband prediction error variance is denoted here by σ_e^2 and the prediction error variance of the m^{th} subband is denoted by $\sigma_{e,m}^2$. Note that these are the same as the ρ_p and $\rho_p^{(m)}$ as defined for the prediction result.

Theorem 5 *For Gaussian AR sources, DPCM coding of subbands offers a gain over DPCM coding of the fullband, for p^{th}-order prediction in the fullband and in each subband, except when the source is white. That is, for non-white sources, the combined first order entropy of the p^{th} order prediction error sequences of the subbands is closer to the entropy rate of the source than is the first order entropy of the p^{th} order prediction error sequence of the fullband source. For white sources, these entropies are equal and there is no DPCM coding gain for subbands.*

Proof: [19]

We can define a prediction error source E with first-order entropy $h_1(\mathbf{E})$ and entropy power Q_e. Recall that for any source X

$$\begin{aligned}\Delta_\infty^X &= h_1(\mathbf{X}) - h_\infty(\mathbf{X}) \\ &= \lim_{N\to\infty} \Delta_N^X \\ &= \lim_{N\to\infty} [h(X_0) - \frac{1}{N} h(X_1, X_2, \ldots, X_N)] \\ &= \lim_{N\to\infty} I(X_0, X_1, \ldots, X_N). \end{aligned} \tag{75}$$

For a Gaussian source X, this resolves to

$$\Delta_\infty^X = \frac{1}{2} \ln \frac{\sigma_X^2}{Q_X}. \tag{76}$$

In particular, the prediction error sequence E is also Gaussian and its difference of entropies is

$$\Delta_\infty^E = \frac{1}{2} \ln \frac{\sigma_e^2}{Q_e}. \tag{77}$$

When the Gaussian source is decomposed into M subbands, the first- order entropy and entropy rate for the error sequence of the m^{th} subband is given by

$$h_1^m(\mathbf{E}) = \frac{1}{2} \ln 2\pi e \sigma_{e,m}^2 \tag{78}$$

and

$$h_\infty^m(\mathbf{E}) = \frac{1}{2} \ln 2\pi e Q_e^m, \tag{79}$$

where Q_e^m is the entropy power of the equivalent error source and $\sigma_{e,m}^2$ is the prediction error variance of the m^{th} subband. The corresponding difference of error entropies for the m^{th} subband is

$$^{\mathbf{e}}\Delta_\infty^{(m)} = \frac{1}{2} \ln \frac{\sigma_{e,m}^2}{Q_e^m}. \tag{80}$$

The composite difference for the M subbands is then given by

$$\begin{aligned} ^{\mathbf{e}}\Delta_\infty^{SB} &= \sum_{m=1}^{M} \frac{^{\mathbf{e}}\Delta_\infty^{(m)}}{V_m} \\ &= \sum_{m=1}^{M} \frac{1}{2V_m} \ln \frac{\sigma_{e,m}^2}{Q_e^m} \\ &= \frac{1}{2} \left(\sum_{m=1}^{M} \frac{1}{V_m} \ln V_m \sigma_{e,m}^2 - \frac{1}{2\pi} \int_{I_m} \ln S_e(\omega) d\omega \right). \end{aligned} \tag{81}$$

This can be rewritten as

$$ {}^{\mathbf{e}}\Delta_{\infty}^{SB} = \frac{1}{2} \ln \frac{\prod_{m=1}^{M} (V_m \sigma_{e,m}^2)^{\frac{1}{V_m}}}{Q_e}. \tag{82} $$

Therefore,

$$ \Delta_{\infty}^{E} -^{\mathbf{e}} \Delta_{\infty}^{SB} = \frac{1}{2} \ln \frac{\sigma_e^2}{\prod_{m=1}^{M} (V_m \sigma_{e,m}^2)^{\frac{1}{V_m}}}. \tag{83} $$

Using the Arithmetic-Geometric Mean inequality in the denominator, we arrive at

$$ \Delta_{\infty}^{E} -^{\mathbf{e}} \Delta_{\infty}^{SB} \geq \frac{1}{2} \ln \frac{\sigma_e^2}{\sum_{m=1}^{M} \sigma_{e,m}^2}. $$

From Theorem 1, the aggregate subband prediction error variance is less than the the fullband prediction error variance Therefore, $\Delta_{\infty}^{E} - {}^{\mathbf{e}}\Delta_{\infty}^{SB} \geq 0$. Equality holds in the above equation if the source has a white spectrum. ♠

Therefore, DPCM coding of subbands offers a gain over fullband DPCM coding, for finite orders of prediction. It should be pointed out that the proof assumes that the prediction error sources are Gaussian. In practice, the error sequences result from prediction on past reconstructed sequences and are more often assumed to be Laplacian. However, the Gaussian assumption provides an upper bound to all other sources with the same variance. This theorem can be easily generalized for vector DPCM coding, when the prediction order exceeds the vector dimension.

6 "Super-Optimal PEF"

In the previous section, we showed that the composite sfm of the subbands is greater than the sfm of the fullband. This result is significant, since it points to the existence of a PEF better than the so-called "optimal" PEF obtained from the solution of the fullband Yule-Walker equations. A PEF is essentially a filter that attempts to flatten the input spectrum. For whiter input spectra, one would intuitively expect a lower order of prediction, or equivalently, a smaller number of AR parameters to adequately predict the process. Therefore, since the subbands have a combined spectrum that is whiter than the fullband spectrum, one would expect a "more optimal" PEF from the subbands. Formally,

Result 1 *[19] For a W.S.S process with spectrum $S_x(\omega)$, there exists a PEF $\tilde{A}^{(SB)}(\omega)$, obtained by operations on the subbands, that is superior (in terms of prediction error performance) to the PEF $\tilde{A}(\omega)$ obtained by solving the fullband Yule-Walker equations, for a given order p.*

Recall that we showed earlier that, when the source is an AR(p) source, the (ideal) subbands are of reduced order. The reduction in the m^{th} subband is by a factor of at least V_m. For general continuous sources, this means that a fullband PEF of order

p is equivalent to decomposing this PEF into order at most $(\frac{p}{V_m})$ PEFs in the M subbands, with V_m being the subsampling factor in the m^{th} subband. Evidently, the subbanded versions of the fullband AR(p) model need not be optimal for the subbands. By allocating the optimal $\frac{p}{V_m}$ model for each subband and recombining these optimal reduced order subband PEFs, we should arrive at a "super-optimal" p^{th}-order subband PEF, which is better than the p^{th}-order fullband PEF in a mean-squared error sense.

Result 2 *The minimum p^{th}-order prediction error of the fullband exceeds the aggregate of the p^{th}-order prediction error of the subbands, when the m^{th} subband PEF is of order $\frac{p}{V_m}$. i.e.,*

$$\rho_p \geq \sum_{m=1}^{M} \rho_{p_m}^{(m)} \tag{84}$$

where ρ_p is the minimum p^{th}-order prediction error of the fullband and $\rho_{p_m}^{(m)}$ is the minimum $(\frac{p}{V_m})^{th}$-order prediction error of the m^{th} subband. Equality holds if the source spectrum is white.

Proof:

Let $\mathcal{F} = \{{}^u A(\omega)\}, u \in U$ be the set of all possible PEFs of all orders up to and including p. Let $\{G_{p_m}\}, m = 1, 2, \ldots, M$ be the sets of PEFs with orders $p_m = \frac{p}{V_m}$. Let K_m be the index set for G_{p_m}. Clearly,

$$G_{p_m} \subset \mathcal{F} \quad m = 1, 2, \ldots, M. \tag{85}$$

Therefore, ρ_p, which is the minimum p^{th}-order prediction error of the fullband, can be written as

$$\begin{aligned}
\rho_p &= \min_{v \in U} \frac{1}{2\pi} \int_{-\pi}^{\pi} |{}^v A(\omega)|^2 P_{xx}(\omega) d\omega \\
&= \min_{v \in U} \frac{1}{2\pi} \sum_{m=1}^{M} \int_{I_m} |{}^v A(\omega)|^2 P_{xx}(\omega) d\omega \\
&= \min_{v \in U} \frac{1}{2\pi} \sum_{m=1}^{M} \int_{-\pi}^{\pi} |{}^v A^{(m)}(\omega)|^2 P_{xx}^{(m)}(\omega) d\omega \\
&\geq \frac{1}{2\pi} \sum_{m=1}^{M} \min_{v \in K_m} \int_{-\pi}^{\pi} |{}^v A^{(m)}(\omega)|^2 P_{xx}^{(m)}(\omega) d\omega \\
&= \frac{1}{2\pi} \sum_{m=1}^{M} \int_{-\pi}^{\pi} |\tilde{A}_{p_m}^{(m)}(\omega)|^2 P_{xx}^{(m)}(\omega) d\omega \\
&= \sum_{m=1}^{M} \rho_{p_m}^{(m)}
\end{aligned} \tag{86}$$

The $\{\tilde{A}_{p_m}^{(m)}(\omega)\}$ are the optimal subband PEFs, with the m^{th} subband PEF having an order at most $\frac{p}{V_m}$. Therefore,

$$\rho_p \geq \sum_{m=1}^{M} \rho_{p_m}^{(m)} \tag{87}$$

with equality if the source spectrum is white.

♠

Remarks

- The "super-optimal" p^{th}-order PEF $\tilde{A}^{(SB)}(\omega)$ is obtained as the combination of the M optimal $\frac{p}{V_m}$-order subband PEFs $\tilde{A}^m(\omega)$, $m = 1, \ldots, M$, where V_m is the subsampling factor in the m^{th} subband, and p is the fullband prediction order.

- It should be noted that, although the "super-optimal" PEF can be mathematically expressed as the combination of optimal reduced order subband PEFs, it is not realizable. The optimal reduced order subband PEFs make use of *fractionally* indexed time samples, which are not available. The above result, therefore, only proves the *existence* of a "super-optimal" PEF but does not provide a realizable form for it. The fact that realizable PEFs are constrained to have integer orders provides the motivation for the algorithmic approaches to constructing the reduced order PEFs.

In terms of entropy reduction, we can state the following theorem (analogous to Theorem 2).

Theorem 6 *For Gaussian sources, the composite p^{th} order entropy of the combined subbands is closer to the entropy rate of the source than the p^{th} order entropy of the fullband, i.e.,*

$$\begin{aligned} \Delta\mathcal{H} &= {}^p\Delta_\infty^X - {}^p\nabla_\infty^{SB} \\ &\geq 0 \end{aligned} \tag{88}$$

where ${}^p\Delta_\infty^X$ is the p^{th} order memory of the fullband and ${}^p\nabla_\infty^{SB}$ is the memory of the combined subbands with an aggregate order of p.

Proof :

Let us define $\Delta\mathcal{H}$ as the memory reduction due to subbanding when the order of prediction p in the fullband is equal to the sum of the prediction orders p_m, $m = 1, \ldots, M$, in the M subbands. The p_m^{th} order memory of the m^{th} subband is defined as [3, 15]

$${}^{p_m}\Delta_\infty^{(m)} = \frac{1}{2}\ln\frac{|\Phi_{p_m}^{(m)}|^{\frac{1}{p_m}}}{Q_{SB}^{(m)}} \tag{89}$$

and the p^{th} order memory of the fullband is defined as

$${}^p\Delta_\infty = \frac{1}{2}\ln\frac{|\Phi_p|^{\frac{1}{p}}}{Q_{FB}}. \tag{90}$$

The individual subbands are combined as below

$${}^p\nabla_\infty^{SB} = \sum_{m=1}^{M}\frac{1}{V_m}\,{}^{p_m}\Delta_\infty^{(m)}. \tag{91}$$

Therefore $\Delta\mathcal{H}$ is given by

$$\Delta\mathcal{H} = \frac{1}{2}\ln\frac{|\Phi_p|^{\frac{1}{p}}}{\prod_{m=1}^{M}(V_m|\Phi_{p_m}^{(m)}|^{\frac{1}{p_m}})^{\frac{1}{V_m}}} \tag{92}$$

Recall that $|\Phi_p| = \prod_{i=0}^{p-1}\rho_i$, the above equation can be written as

$$\Delta\mathcal{H} = \frac{1}{2}\ln\frac{(\prod_{i=0}^{p-1}\rho_i)^{\frac{1}{p}}}{\prod_{m=1}^{M}(V_m(\prod_{i=0}^{p_m-1}\rho_i^{(m)})^{\frac{1}{p_m}})^{\frac{1}{V_m}}}. \tag{93}$$

From the previous theorem,

$$\rho_i \geq \sum_{m=1}^{M}\rho_{i_m}^{(m)} \tag{94}$$

for $\sum_{m=1}^{M} i_m = i$. Following the proof of Theorem 3,

$$\begin{aligned}\Delta\mathcal{H} &\geq \frac{1}{2}\ln\frac{\prod_{i=0}^{p-1}(\sum_{m=1}^{M}\rho_{i_m}^{(m)})^{\frac{1}{p}}}{\prod_{m=1}^{M}(V_m(\prod_{i=0}^{p_m-1}\rho_i^{(m)})^{\frac{1}{p_m}})^{\frac{1}{V_m}}}\\ &\geq \frac{1}{2}\ln\frac{\prod_{i=0}^{p-1}(\sum_{m=1}^{M}\rho_{i_m}^{(m)})^{\frac{1}{p}}}{\sum_{m=1}^{M}\prod_{i=0}^{p_m-1}(\rho_i^{(m)})^{\frac{1}{p_m}}}.\end{aligned} \tag{95}$$

Once again, the argument of the logarithm is a ratio of product of sums to sums of products, with all non-negative terms. Since the numerator includes all the terms of the denominator, $\Delta\mathcal{H} \geq 0$.

♠

We have showed the existence of a "super-optimal" PEF that has the same order as the optimal PEF obtained from solving the fullband Yule-Walker equations. This "super-optimal" PEF $\tilde{A}^{(SB)}(\omega)$ effectively acts on the source directly, though, of course, the optimal PEF for each subband can only be obtained by operations on each subband. This leads directly to the question: What is an optimal subband PEF? In other words, given an order p, how can we find p_i for the i^{th} subband such that the combination of the M subband PEFs results in the best PEF? We propose to answer this question by analogy with the well-known subband rate-allocation problem.

6.1 Subband Order Allocation

Since each subband has a different sfm, one would intuitively choose a lower order of prediction for the flatter subbands, and conversely. We can formulate the following problem:

Given p, the order of prediction, and M, the number of subbands, find the optimal assignment of orders p_i, $\sum_{i=1}^{M} p_i = p$, such that the composite sfm of the error spectrum is closest to 1. Equivalently, we can state this problem as :

minimize $\prod_{m=1}^{M} \rho_{p_m}^{(m)}$, subject to the constraint that $\sum_{m=1}^{M} p_m = p$, with p_m being integers.

The $\rho_{p_m}^{(m)}$'s are the prediction error powers and they are a decreasing function of the prediction order. This second formulation can be shown to be equivalent to the first one as follows

$$\begin{aligned}
\xi_e^{SB} &= \prod_{m=1}^{M} \left(\xi_e^{(m)}\right)^{\frac{1}{V_m}} \\
&= \prod_{m=1}^{M} \left(\frac{\xi_x^{(m)}\sigma_m^2}{\rho_{p_m}^{(m)}}\right)^{\frac{1}{V_m}} \\
&= \xi_x^{SB} \frac{\prod_{m=1}^{M}(\sigma_m^2)^{\frac{1}{V_m}}}{\prod_{m=1}^{M}(\rho_{p_m}^{(m)})^{\frac{1}{V_m}}} \qquad (96)
\end{aligned}$$

Since the composite sfm of the subbands and the subband variances are fixed, the maximization of the sfm of the prediction error spectrum is equivalent to minimization of the geometric mean of the prediction error powers. The corresponding "more optimal" PEF will be given by

$$A^{SB}(\omega) = \sum_{m=1}^{M} H^m(\omega) A_{p_m}^m(\omega V_m) \qquad (97)$$

where $H^m(\omega)$ is the m^{th} bandpass filter and $A_{p_m}^m(\omega)$ is the optimal PEF of order p_m for the m^{th} subband. The resemblance to the rate-allocation problem encountered in subband coding is clear. The problem in subband coding is that of optimally assigning the rate among the subbands in order to minimize the distortion. The key difference is in the fact that it is not possible to solve explicitly for the prediction order. This is because of the fact that the prediction order is implicitly present in the autocorrelations. In the following section, we present two algorithms for order assignment which are optimal for the stated constraints.

6.2 Algorithms

In this section, we describe two algorithms to determine the optimal subband prediction order. Figs. 6 and 7 show the flowcharts for these algorithms.

6.2.1 Algorithm 1

The constraints we assume are: (a) The total order of prediction is fixed at p. (b) The total number of subbands should not exceed M.

1. Split fullband into 2 equal subbands.
2. Order-$p/2$ prediction in each subband.
3. Is the total number of subbands equal to M ? If so, stop.

4. Compute the Geometric Mean and Arithmetic Mean of the prediction errors. The ratio G.M/A.M is a measure of the subband flatness.

5. Order the sfm's in decreasing order. Note that the sfm is computed only for the newly created subbands but the ordering is over all subbands.

6. Split the subband with lowest sfm.

7. Order-$p/4$ prediction in each of the two new subbands.

8. Proceed until the number of subbands reaches M or until the sfms achieve a desired flatness.

This approach is the best possible for a binary tree-structured subbanding. The prediction errors are obtained from the solution of the Yule-Walker equations for each subband.

6.2.2 Algorithm 2

The constraints are: (a) Fixed total order of prediction p and (b) no constraint on the number of subbands M.

1. Split fullband into 2 equal subbands.

2. Order-1 prediction in each new subband.

3. Check if total prediction order exceeds p. If so, stop.

4. Split the subband with the greatest prediction error.

5. Go to 2

This algorithm will yield p subbands with order 1 prediction in each subband. However, the subbands are not uniform, but octave- related over several levels. This algorithm assigns a greater frequency resolution wherever the spectral activity is higher. The prediction error is determined by

$$\rho_p = \sigma^2(1 - |a_1[1]|^2) \tag{98}$$

All the parameters in the above equation are computed for each subband. σ^2 is the variance of each subband and $a_1[1]$ is the order-1 AR parameter for that subband. At each iteration, the prediction error is computed for the two new subbands and ordered with respect to the previously computed prediction errors.

6.3 AR Modeling and Linear Prediction

The Linear Prediction equations are structurally identical to the Yule-Walker equations of an AR process [16, 13]. The distinction lies in the interpretation of the equations and the application. The AR filter usually receives a white noise input sequence $u(n)$ of variance σ^2 and generates an output AR sequence $x(n)$, viz.,

$$x(n) = -\sum_{k=1}^{p} a(k)x(n-k) + u(n) \tag{99}$$

The linear PEF observes an input sequence $x(n)$, makes a linear estimate

$$\hat{x}(n) = -\sum_{k=1}^{p} \alpha(k)x(n-k) \tag{100}$$

and generates an output error sequence $e(n)$.

$$\begin{aligned} e(n) &= x(n) - \hat{x}(n) \\ & x(n) + \sum_{k=1}^{p} \alpha(k)x(n-k) \end{aligned} \tag{101}$$

The coefficients $\{\alpha(i)\}$ are chosen to minimize the prediction error variance

$$\rho = \mathrm{E}[|\hat{x}(n) - x(n)|^2]. \tag{102}$$

This minimization results in the Wiener-Hopf equations [16]

$$r_{xx}(k) = -\sum_{l=1}^{p} \alpha(l) r_{xx}(k-l) \quad k = 1, \ldots, p \tag{103}$$

which are identical to the Yule-Walker equations obtained for an AR(p) model. Therefore, the coefficients $\alpha(i) = a(i), i = 1, \ldots, p$ and the prediction error variance $\rho = \sigma^2$. The error sequence $e(n)$ is white only if the input sequence $x(n)$ is an AR(p) process *and* the p^{th}-order PEF is optimal. In this sense, the PEF can be viewed as a whitening filter. When AR models are fitted to a source, the AR filter does precisely the same - it attempts to whiten the input sequence.

Therefore, in AR spectral modeling applications, the AR filter is identical to the linear PEF. This equivalence allows all the linear prediction results proved earlier to be applied to AR spectral estimation. In particular, the following prediction results can be rephrased for AR spectral estimation:

- The minimum prediction error from p^{th}-order fullband AR modeling exceeds the aggregate of the minimum prediction error from p^{th}-order AR modeling in each subband.

- The minimum prediction error from p^{th}-order fullband AR modeling exceeds the aggregate of the minimum prediction error over all subbands from $\frac{p}{V_m}$-order AR modeling in the m^{th} subband.

In other words, in proving the superior error performance of linear prediction from subbands, we have shown the same for AR modeling from subbands.

7 Simulation Results

An AR(15) model was used to generate the source data, which was then subbanded into 2 and 4 equal subbands, which correspond to levels 1 and 2 of a pyramid decomposition of the source. AR spectral estimates of order 3 were made in the fullband and in each of the decimated subbands. The subband power spectral densities were upsampled, filtered, and summed to reconstruct the fullband spectrum. Let the spectral estimate of the m^{th} subband be denoted by $P^{(m)}(\omega)$. The re-filtering is effected by passing this estimate through the corresponding reconstruction filter $H^m(\omega)$ to obtain the re-filtered estimate $P^{(m)}(\omega)|H^m(\omega)|^2$. This estimate is then up-sampled by the appropriate up-sampling factor V_m. The re-filtered, up-sampled spectral estimate of the m^{th} subband is now referenced to the fullband frequency axis. This procedure is carried out for all M subbands and the resulting sum is referred to as the reconstructed subband spectral estimate. The subband decomposition was effected by using Johnston's 32C filters [14]. These are Quadrature-Mirror Filters (QMFs) and are not perfect reconstruction filters. In order to make them perfect reconstruction filters, the original tap values were modified to provide (almost) zero reconstruction error. The resulting modified 32C filters are close to perfect reconstruction filters.

The mean-square-error (mse) between the spectrum of the AR(15) source and the AR(3) fullband estimate was 77.1 while the mse between the source spectrum and the reconstructed level-1 AR(3) (two subbands) subband spectra was 60.22. This is an improvement of about 1.1 dB. When we use four subbands, the mse drops to 3.88 or about 13 dB better than the fullband estimate. The results, plotted in Figs. **??**, 9, and10, show the greater resolution and accuracy of the subband spectral estimates.

A second set of AR spectral estimates was made, with the constraint that the total number of parameters used for the fullband and all the subbands were the same. An AR(12) model was fitted to the fullband data and AR(3) models were fitted to each of the four level-2 subbands (see Figs. 11 and 10). The mse for the AR(12) fulband estimate was 6.9633 while for the reconstructed level-2 subband spectrum it was 3.88. i.e., a gain of 2.2 dB. The mse improvement does not give a true indication of the superiority of the subband estimate. As can be seen from comparing Fig. 11 and Fig. 10, the subband estimate has far better resolution and is able to pick out the important features in the source spectrum.

Note that for the second set, each subband was assigned the same order. This is a sub-optimal assignment, since it makes no use of the fact that each subband has a different sfm.

8 Conclusion

Theoretical justification for the use of finite-order subband operations has been provided through the derivation of several gain theorems deduced by investigation of the spectral properties of subbands of a source. The gains obtainable from p-th order subband prediction and vector p-tuple coding have been quantified in formulas. The gain of subband DPCM over fullband DPCM, for finite orders of prediction, has also been provided in a formula. The weakness of the gain theorems is that it could not be proved that there is always a strictly positive gain for subband operations on a non-white source. The restriction to ideal subbands produced by unrealizable, infinitely sharp (brickwall) filters certainly aided the analysis, but also indicates the extension to non-ideal, realizable finite- order filters. Recall Corollary (4), which states that the memory reduction $\Delta\mathcal{H}$ due to ideal subbanding decreases to zero as the entropy or prediction order p increases without limit. It has been found in [22] and [23] that there is a performance loss for optimal (infinite-order) coding or prediction in subbands produced by paraunitary filter banks. The reason is that the composite p-th order memory of the subband is only less than that of the fullband up to a certain order of p and no greater, due to the (uncancellable) aliasing power of non-ideal subbands. As shown in [21] for paraunitary filter banks, that value of p is found by setting the subband memory reduction $\Delta\mathcal{H}$ in (18) to 0, wherein the determinants of the covariance matrices are those for non-ideal subbands.

The equivalence of Linear Prediction and AR modeling justifies the use of subbands in spectral estimation problems and simulation results have been presented. The results clearly indicate the superior performance of spectral estimation from subbands. They also imply positive gains for vector coding, linear prediction, and DPCM in subbands of the same sources. The memory reduction properties inherent in subbands lead also to the faster convergence of subband adaptive filters over fullband adaptive filters in adaptive line enhancers and adaptive channel equalizers [24, 25]. The existence of a "super-optimal" Prediction Error Filter has been proved and a framework has been provided to further investigate this issue. Finally, two algorithms have been presented for the design of the "super-optimal" PEF.

References

[1] J. W. Woods and S. D. O'Neil, "Subband Coding of Images," *IEEE Trans. Acoustics, Speech, and Signal Processing*, vol. ASSP-34, pp. 1278–1286, Oct. 1986.

[2] A. D. Wyner and J. Ziv, "Bounds on the Rate-Distortion Function for Stationary Sources with Memory," *IEEE Trans. Information Theory*, vol. IT-17, pp. 508–513, Sept. 1971.

[3] W. A. Pearlman, "Performance Bounds for Subband Coding," Chap. 1, *Subband Image Coding*, J. W. Woods, Ed., Kluwer Academic Pulishers, 1991.

[4] *Subband Image Coding*, J. W. Woods, Ed., Kluwer Academic Publishers, 1991.

[5] A. Gersho, "Advances in Speech and Audio Compression," *Proceedings of the IEEE*, vol. 82, pp. 900–918, June 1994.

[6] R. P. Rao and W. A. Pearlman, "On Entropy Rate of Pyramid Structures," *IEEE Trans. Inform. Theory*, vol. 37, pp. 407-413, March 1991 .

[7] J. D. Markel and A. H. Gray, Jr., *Linear Prediction of Speech*, Springer-Verlag, New York, 1976.

[8] J. I. Makhoul and J. J. Wolf, " Linear Prediction and Spectral Analysis of Speech," Bolt, Beranek and Newman, Inc., Technical Report, 1972.

[9] M. B. Priestley, *Spectral Analysis and Time Series*, Academic Press, 1989.

[10] S. M. Kay, *Modern Spectral Estimation: Theory and Applications*, Prentice-Hall Inc. .

[11] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, 1961.

[12] U. Grenander and G. Szego, *Toeplitz Forms and their Applications*, University of California Press, Berkeley and Los Angeles, CA and Cambridge University Press, London, England, 1958.

[13] S. L. Marple, *Digital Spectral Analysis with Applications*, Prentice-Hall Inc., 1987.

[14] N. S. Jayant and P. Noll, *Digital Coding of Waveforms*, Prentice-Hall Inc., 1984.

[15] Sudhakar Rao and W. A. Pearlman, " On the superiority of coding and estimation from subbands," Proc. 1992 Conference on Information Sciences and Systems, Princeton, N.J., pp. 890–895, March 1992.

[16] A. Papoulis, "Maximum Entropy and Spectral Estimation: A Review," IEEE Trans. Acoustics, Speech and Signal Processing, vol. 29, pp. 1176–1186,Dec. 1991.

[17] M. Basseville *et al.*, "Modeling and Estimation of Multiresolution Stochastic Pprocesses," *IEEE Trans. Inform. Theory*, vol. 38, pp. 766–784, March 1992.

[18] M. Basseville *et al.*, "Multiscale Autoregressive Processes, Part I: Schur-Levinson Parameterizations", *IEEE Trans. Signal Processing*, vol. 40, pp. 1915–1934, Aug. 1992.

[19] Sudhakar Rao and W.A. Pearlman, " Spectral Estimation from Subbands," Proc. IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis, pp. 69–73, Victoria, Canada, 1992.

[20] Sanjiv Nanda and W.A. Pearlman, " Tree Coding of Subbands," *IEEE Trans. Image Processing*, Vol. 1, pp. 133–147, April 1992.

[21] S. Rao, *Analysis of Linear Prediction and Spectral Estimation from Subbands with Applications*, Ph.D. Thesis, Electrical, Computer and Systems Engineering Dept., Rensselaer Polytechnic Institute, Dec. 1993.

[22] T. R. Fischer, "On the Rate-Distortion Efficiency of Subband Coding," *IEEE Trans. Information Theory*, vol. IT-38, pp. 407–413, Mar. 1992.

[23] S-L. Tan and T. R. Fischer, "Liner Prediction of Subband Signals," *IEEE J. Selected Areas in Commmunications*, vol. 12, pp. 1576-1583, Dec. 1994.

[24] Y. G. Yang, N. I. Cho, and S. U. Lee, "On the Performance Analysis of the Subband Adaptive Digital Filter," Proc. IEEE Int. Symp. on Circuits and Systems, Vol. I, pp. 64–67, Chicago, IL, May 1993.

[25] S. R. Rao and W. A. Pearlman, "Channel Equalization from Subbands," Proc. 1994 Conf. on Information Sciences and Systems, pp. 431–436, Princeton, NJ, March 1994.

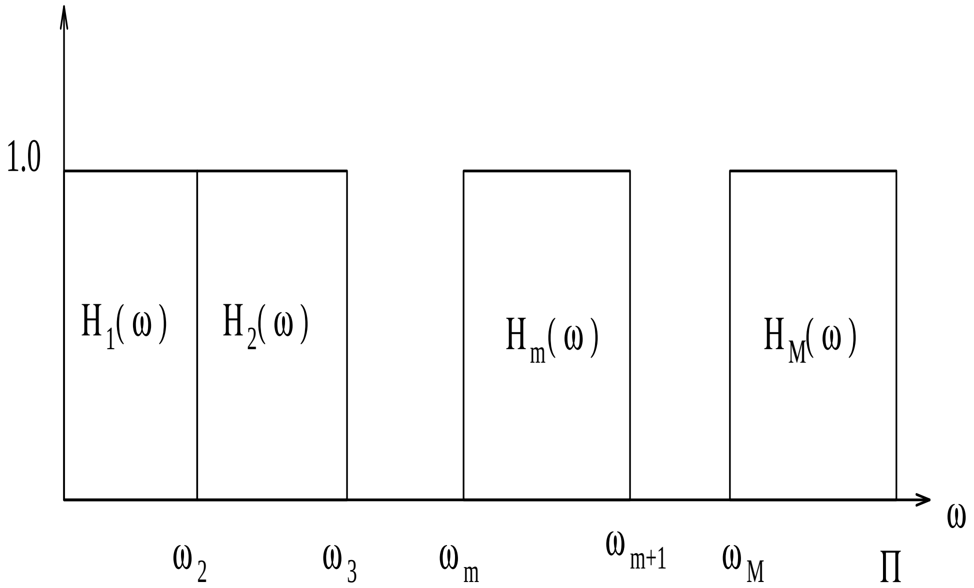


Figure 1: Ideal Bandpass filters

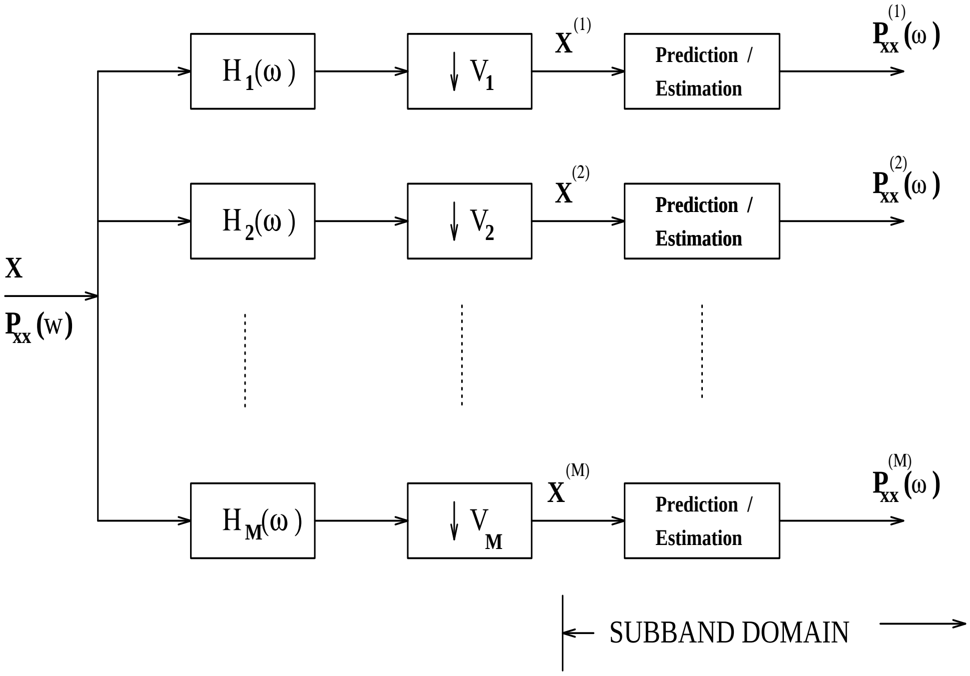


Figure 2: Analysis Filterbank

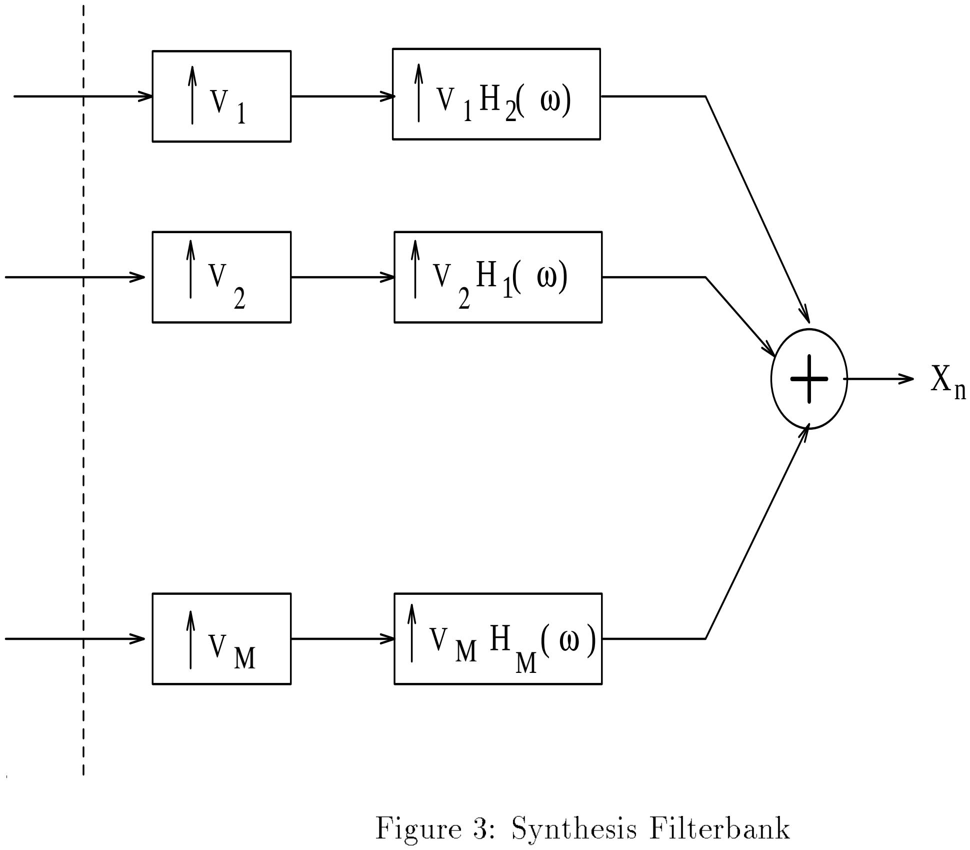


Figure 3: Synthesis Filterbank

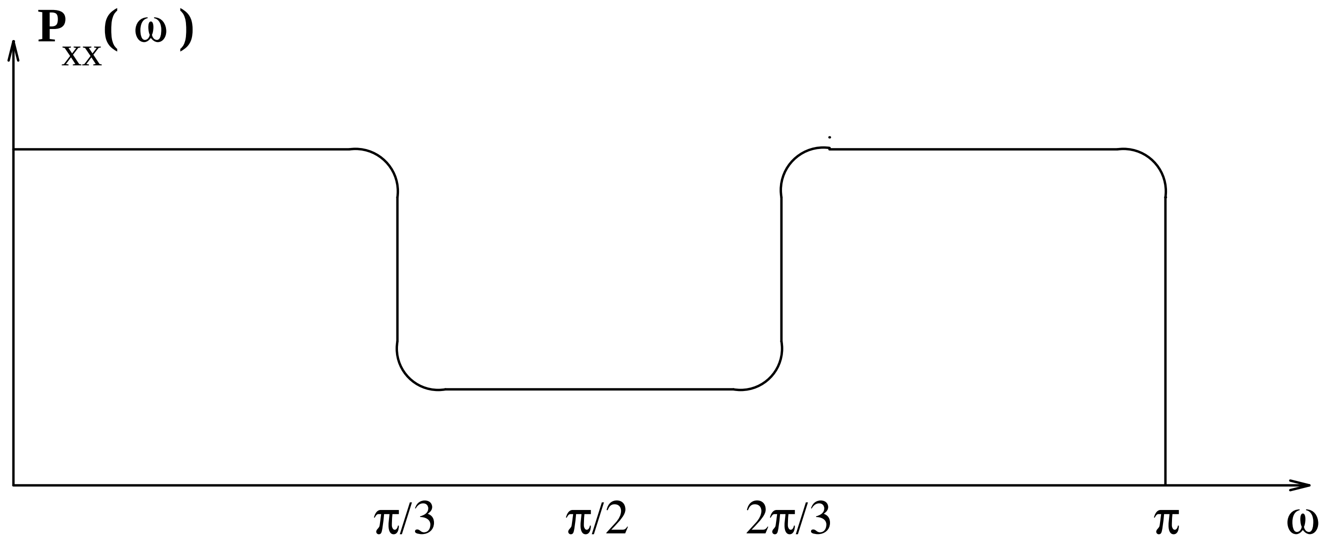
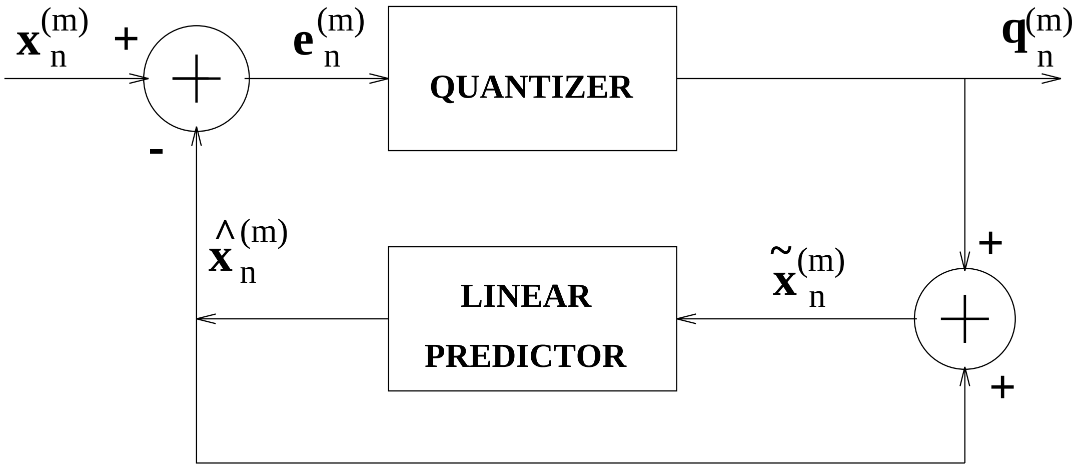
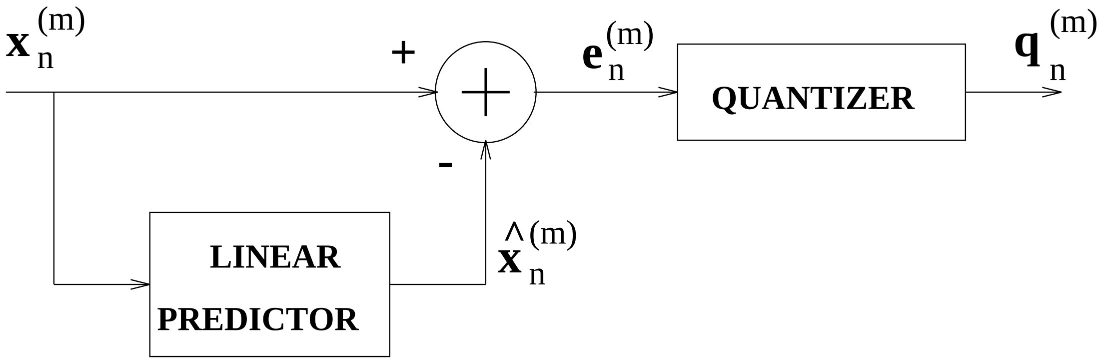


Figure 4: Optimal Subbanding



(a) Practical DPCM Coder for the mth subband



(b) mth subband DPCM Coder for analysis.

Figure 5: DPCM Coders-Theoretical and Practical

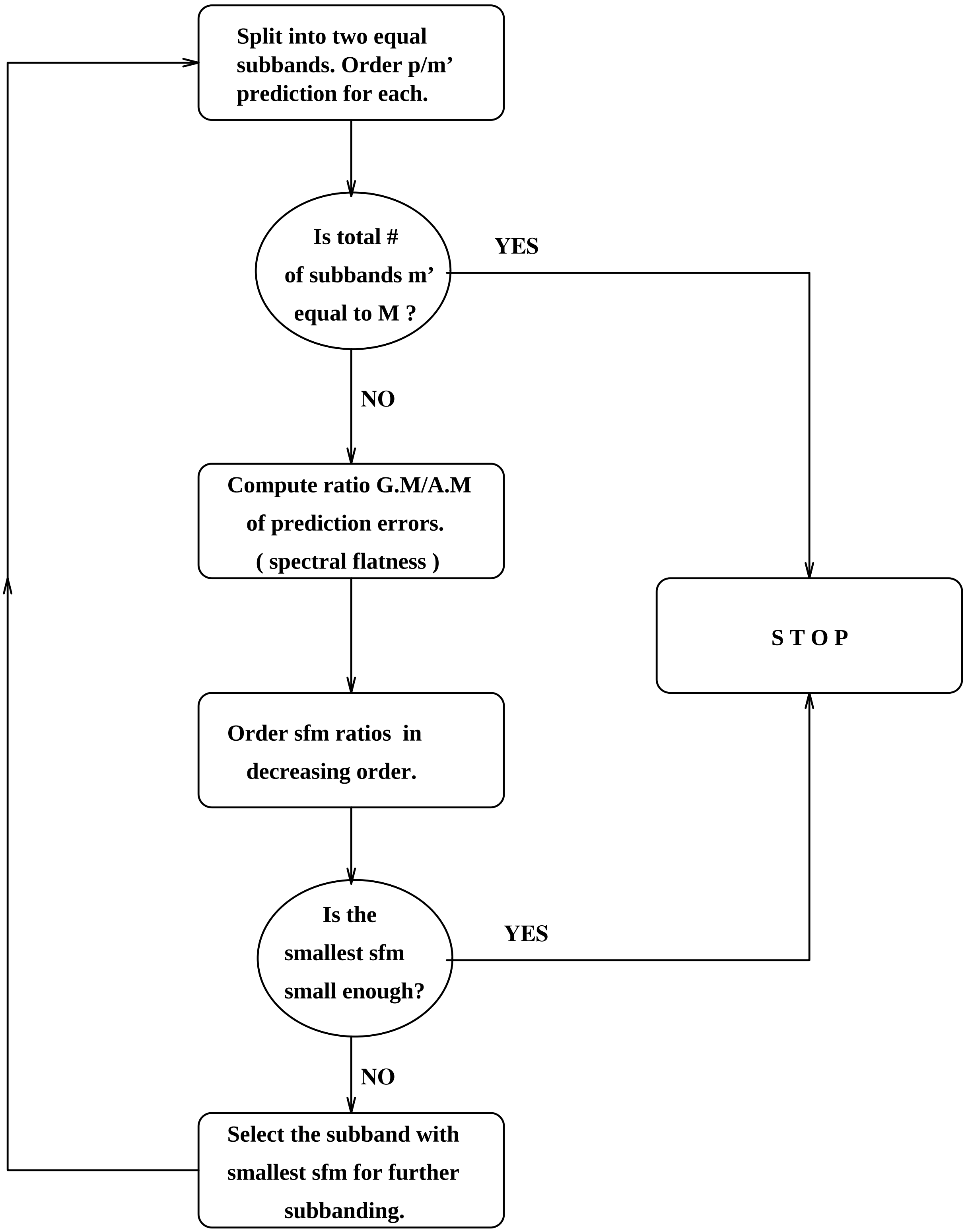


Figure 6: Algorithm 1: Flowchart

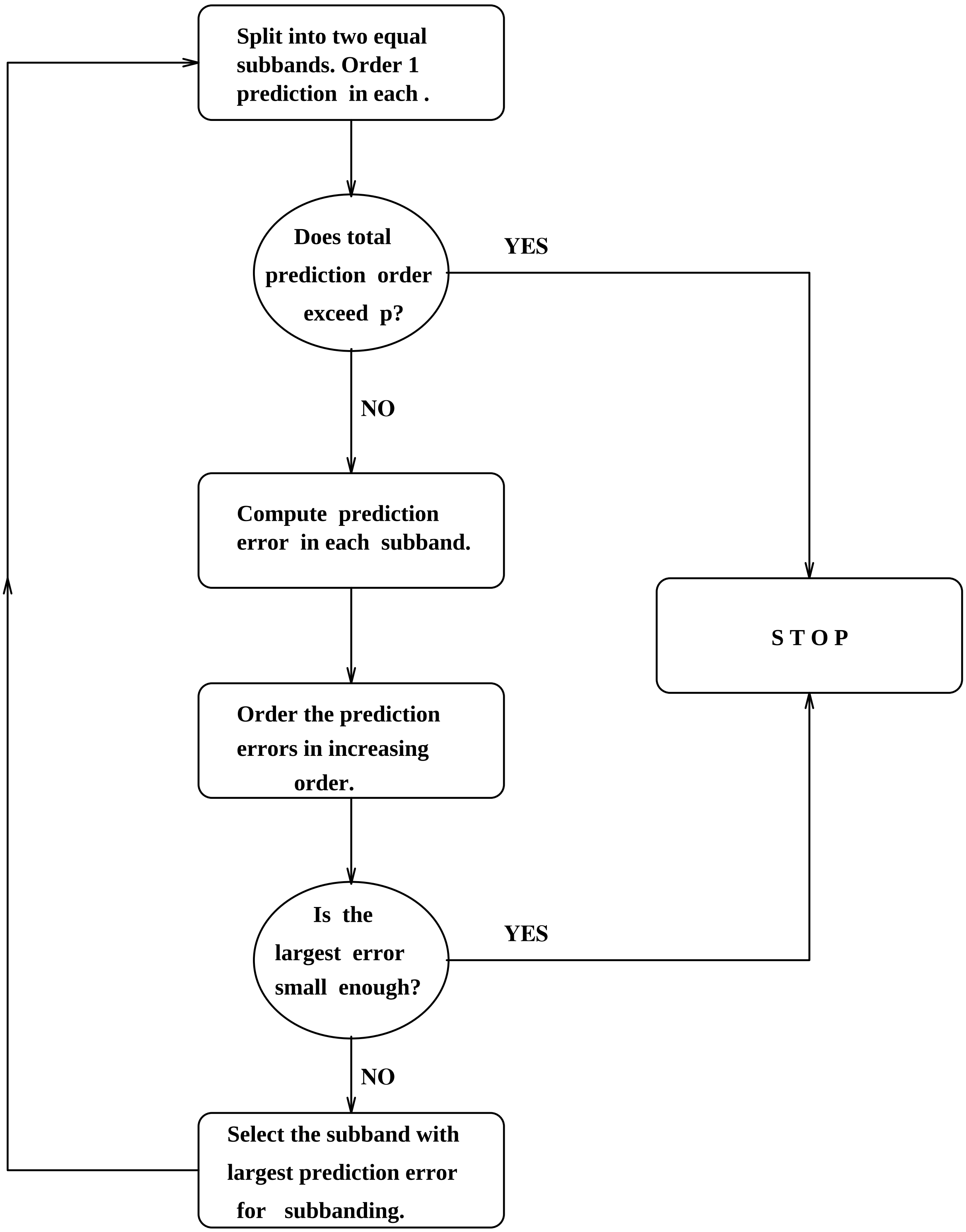


Figure 7: Algorithm 2: Flowchart

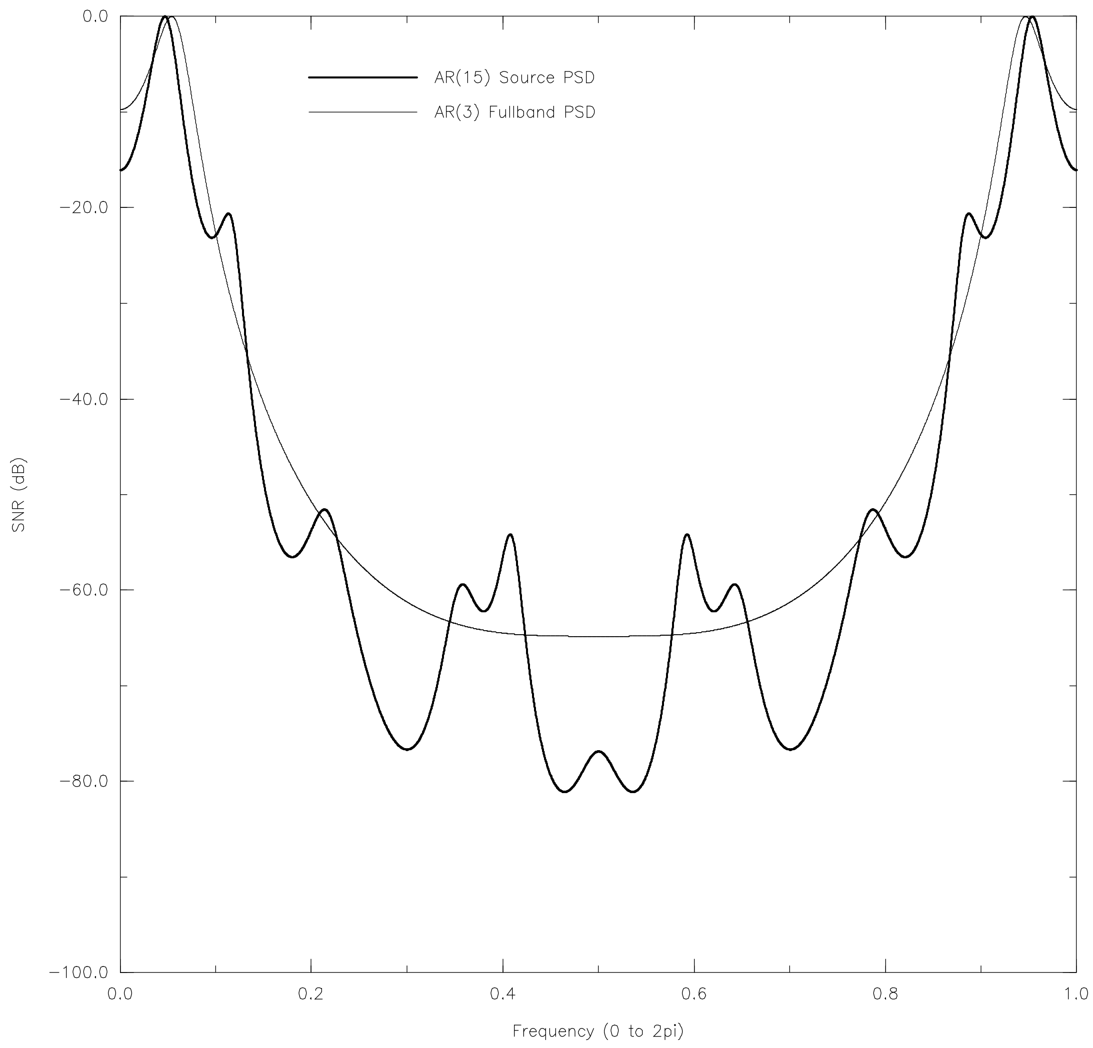


Figure 8: AR(15) Source psd and AR(3) Fullband psd

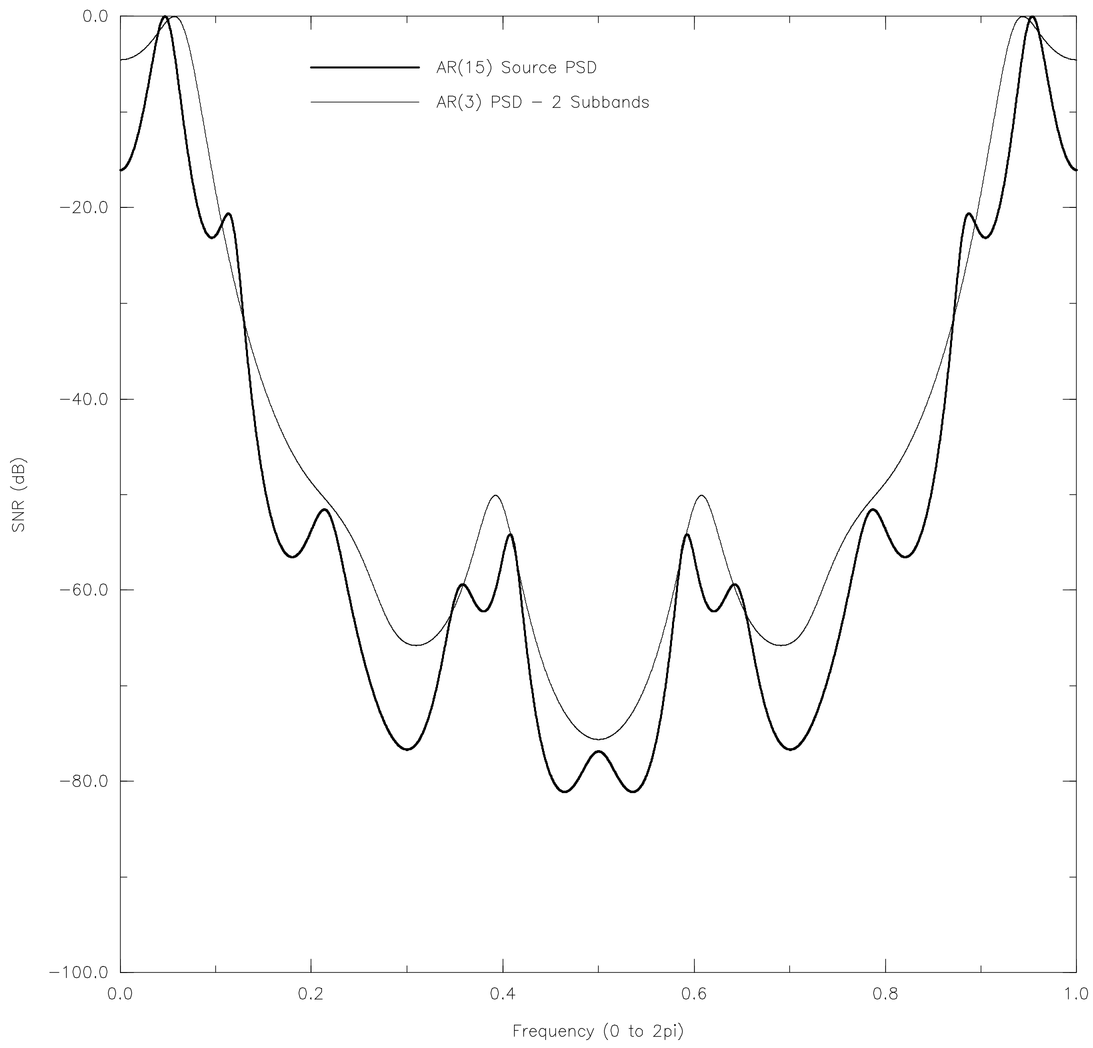


Figure 9: AR(15) Source psd and AR(3) Subband psd: 2 subbands

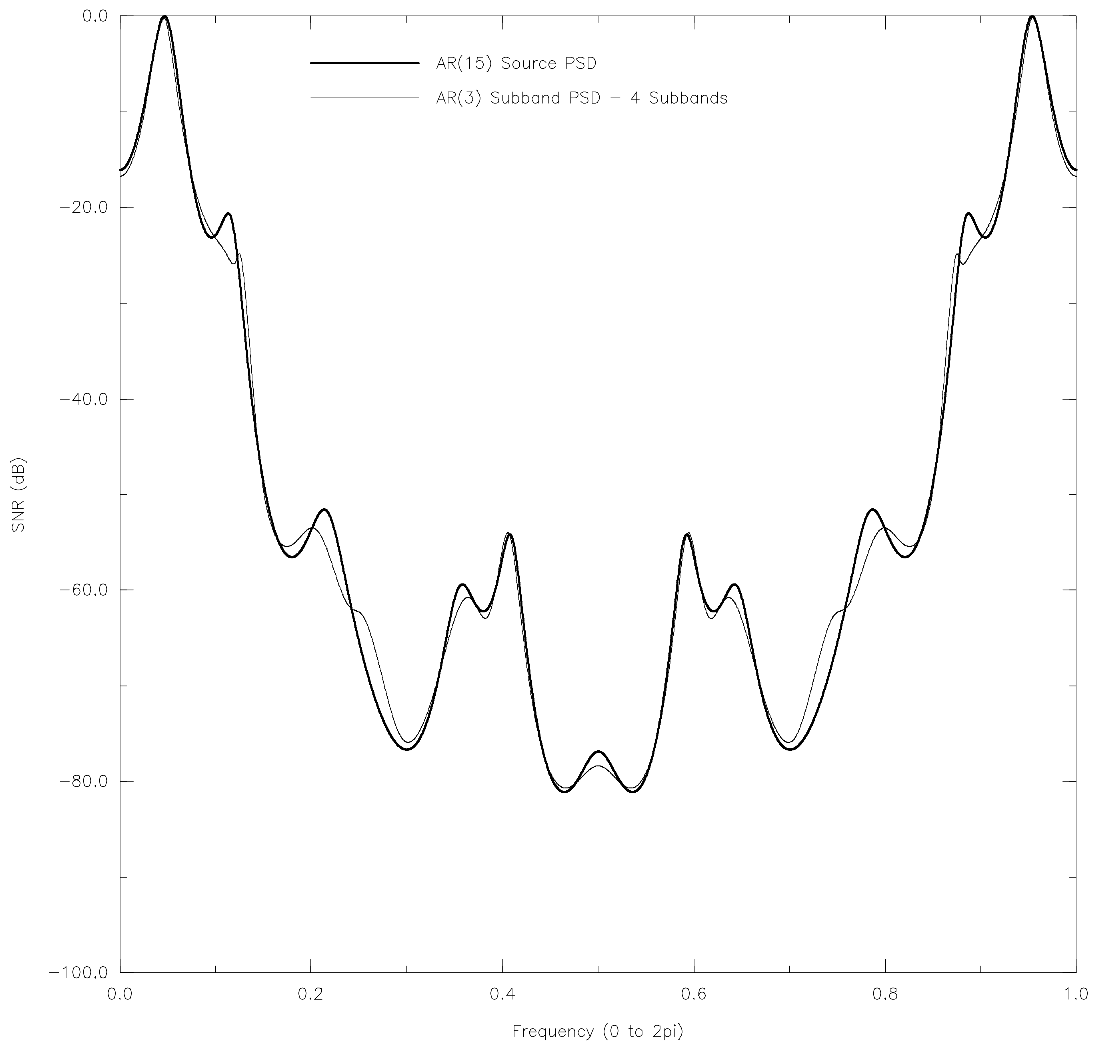


Figure 10: AR(15) Source psd and AR(3) Subband psd: 4 subbands

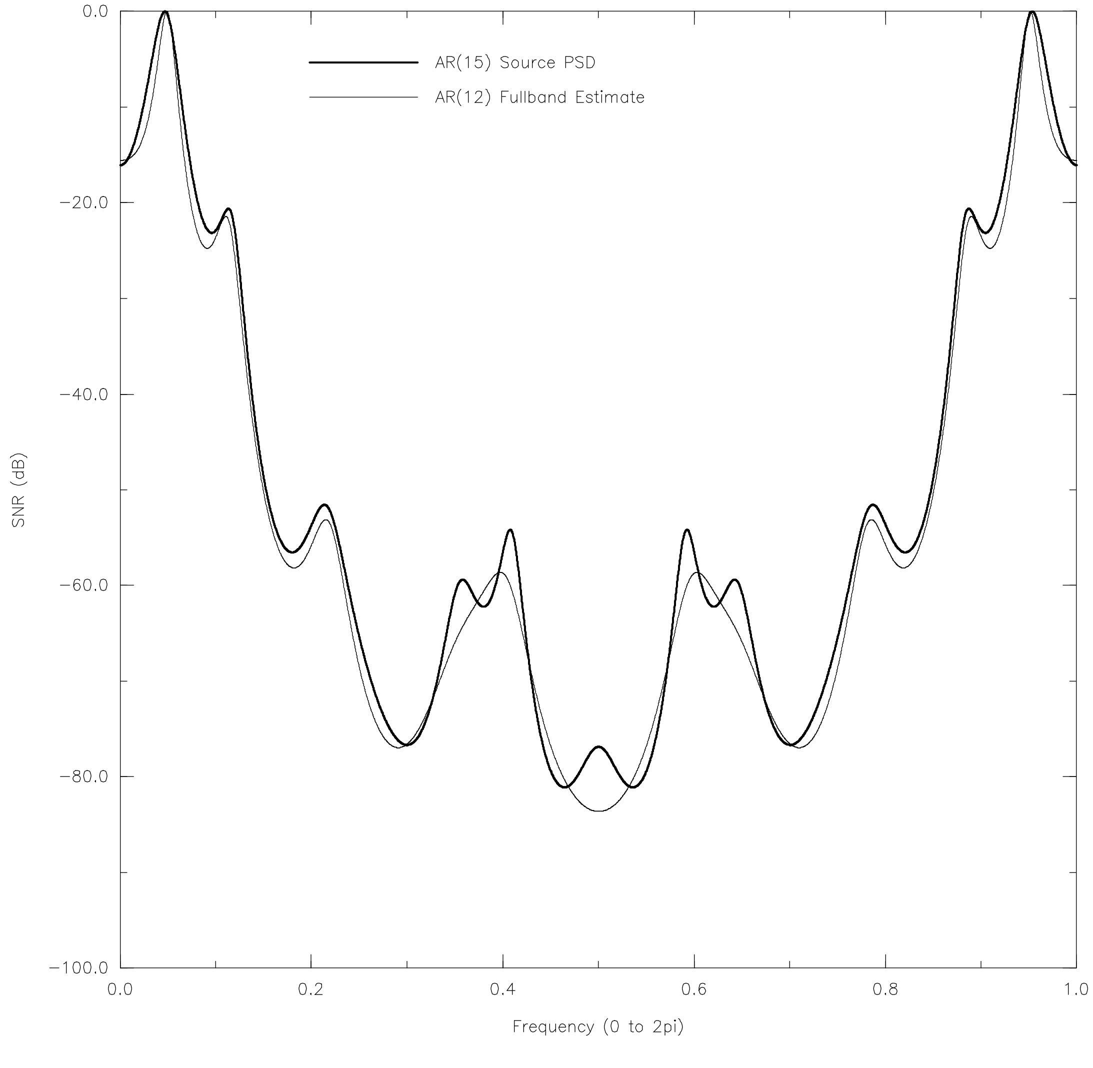


Figure 11: AR(15) Source psd and AR(12) Fullband psd