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Filters and Filter Banks for Periodic Signals, the Zak Transform and Fast Wavelet Decomposition *

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Abstract

In this paper we present a new approach to filtering and reconstruction of periodic signals. The tool that proves to handle these tasks very efficiently is the discrete Zak transform. The discrete Zak transform can be viewed as the discrete Fourier transform performed on the signal blocks. It also can be considered the polyphase representation of periodic signals. Fast filtering-decimation-interpolation-reconstruction algorithms are developed in the Zak Transform domain both for the undersampling and critical sampling cases. The technique of finding the optimal biorthogonal filter banks, i.e. those that would provide the best reconstruction even in the undersampling case, is presented. An algorithm for orthogonalization of non-orthogonal filters is developed. The condition for perfect reconstruction for the periodic signals is derived. The generalizations are made for the non-periodic sequences and several ways to apply the developed technique to the non-periodic sequences are considered. Finally the developed technique is applied to recursive filter banks and the discrete wavelet decomposition.

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1 Introduction

Considerable effort has been made to find good subband filtering algorithms that allow perfect reconstruction of signals, e.g. see [6], [7], [19], [13], [14]. In recent years the development of wavelet theory gave rise to new types of perfect reconstruction filter banks. The wavelet basis function upon which the signals or images are decomposed can be orthogonal or biorthogonal. The orthogonality condition leads to tight restrictions on the filter impulse response see [13]. While biorthogonal wavelets mitigate this drawback to a certain degree, one must still go through a careful process of choosing the appropriate fiters (see [8] and [9]).

In this paper we present a new type of linear filters that can do the same job as filters considered in the above references, i.e., subband decomposition, decimation, interpolation and perfect reconstruction. There are two advantages that these filters possess over other types of filters:

1. They do not have such complicated restrictions on their impulse response as other filters. As a matter of fact, the only restriction for them is that their impulse responses be periodic.

2. The time of filtering and reconstruction is always proportional to a value that is less than $L \log L$, where L is the size of the signal and does not depend on the size of the support of the impulse response of the filters as in the case of QMF filters (though in case the filters are short, one can perform the decomposition and reconstruction in the time domain).

The design of the new filter is based on the properties of the discrete Zak transform. The introduction and historical development of the continuous Zak trensform was presented in [1]. The discrete Zak transform according to the authors' knowledge was first introduced by Auslander, Gertner and Tolmieri in [2]. We believe that the discrete Zak transform deserves no less attention than the discrete Fourier transform in signal analysis.

There is a close similarity between the discrete Zak transform and the polyphase representation. For the definition and properties of the polyphase representation see [6], [7]. One can conceive of the Zak Transform as the polyphase representation in the case of periodic signals. We emphasize this similarity, and, with the help of the Zak Transform are acquiring new insights into the theory of filter banks.

We present also a technique for finding biorthogonal synthesis filter banks for analysis filter banks which have a periodic impulse response, but are otherwise arbitrary. The fact that we can choose periodic filters with arbitrary response is the main advantage of our method of finding the biorthogonal filter pairs over those considered in [6], [7], [8] and [9]. Also our biorthogonal filter technique can be extended to the non-periodic case, since "almost biorthogonal" non-periodic filter banks may be obtained by an extension of a period.

Another advantage of our approach is that it allows us to find the optimal biorthogonal filter bank in case of undersampling, i.e. the case when the decimation factor is larger than the number of the filters, which sometimes may be useful, for example, in coding. Moreover, our algorithm for finding the biorthogonal filter bank is fast and can be implemented in real time, which is quite desirable for some applications.

Filter banks designed with perfect reconstruction filters were used for multiresolution analysis, see [6], [7], [8], [19], [13], [15]. We also present a multiresolution fast filter bank implementation in the Zak transform domain. The proposed decomposition algorithm is the same as the algorithm proposed by Vetterli in [8] for given fixed analysis-synthesis filter banks with non-periodic responses. However, we also present a fast reconstruction algorithm for any type of filter.

Getz in [11] considered a circular wavelet transform. However, he considered only time domain algorithms and only the case when the biorthogonal wavelets are assumed to be known. In a certain respect our multiresolution algorithm shares some features of Getz's, because the function becomes "wrapped" when its period is getting smaller than the support of the function. The precise meaning of this is given in the section 6.

Caire et al in [10] also considered the circular wavelet transform and restricted their attention to the orthogonal case. The circular case (i.e., when signal and impulse response are periodic) is prevalent in this paper; however, the generalization to nonperiodic case can be easily made, using the overlap-add or overlap-save convolution methods, see [4].

The paper is organized in the following way. In section 2, the definition and notations of the discrete Zak transform are given, some Zak transform properties are derived, and the filtering and reconstruction algorithms are presented. In section 3, the expressions for biorthogonal filters and filter banks are obtained in both undersampling and critical sampling cases and the fast orthogonalization algorithm for a filter bank is presented. In section 4 the non-periodic case is considered, biorthogonal filters are found, and the theory is applied to image compression.

In section 5, multiresolution properties of the Zak transform are considered. In section 6, the developed theory is applied to decomposition and reconstruction through filter banks. In section 6 the results are summarized and directions are suggested for future research. The proofs to all of the theorems are placed in Appendix.

1.1 Notations

Here we emphasize some notations that are used throughout the paper. Given a discrete function $h[k]$, we denote by $\bar{h}$ the complex conjugate of h. By h^- the function $h[-k]$. Notice that bar and minus are commutative, meaning $\bar{h}^- = \bar{h^-}$.

2 Zak Transform: Definition, Notations and Properties

First, let us review the definition of the discrete Zak transform and derive some useful properties.

Definition 1 *Let $g[k]$ be L-periodic, where $L = NM$. The discrete $N \times M$ Zak transform of the function $g[k]$ is*

$$Z(g)(\frac{n}{N}, \frac{m}{M}) = \sum_{i=0}^{N-1} g[m + iM]e^{-j\frac{2\pi}{N}in}, 0 \leq n < N, 0 \leq m < M. \tag{1}$$

The discrete-time signal $g[k]$ of length L is divided into N blocks each of size M and the discrete Fourier transform of length N is performed blockwise, in that every block is treated as a vector multiplied by the appropriate complex scalar and summed together to produce the M-dimensional Zak transform of g. Then the first argument of the Zak transform (n) specifies the number of the block, and the second one (m) specifies the place of the sample inside the block.

An example showing the relationships of pixels in a signal of period $L = 6$ under the Zak Transform with the number of blocks $N = 3$ and the number of samples in each block $M = 2$ is shown in Figure 1. The arrows show the samples that are being scaled and summed together to produce the corresponding sample of the Zak Transform .

Sometimes it is convenient to have the Zak transform defined over the whole plane of integer pairs (n, m), not only on the rectangle $0 \leq n < N, 0 \leq m < M$. In such cases we assume that the Zak Transform is given by (1) over the whole plane. It can be proved that the values of the Zak Transform over the rectangle will determine its values over the whole plane. Indeed one can see that the extended Zak transform is N-periodic in n. In m the dependency between samples spaced by M is more complex. Note that the complex factor multiplying the samples is independent of m, the position of a sample within a block.

The notation above for the Zak transform was introduced by Auslander, Gertner and Tolmieri in [2]. We shall adhere to it in the paper only when the number of the blocks N is important. However, most of the time we shall be using the abbreviated notation $Z_M(g)(n,m)$, where M is the size of a block, assuming that the length of a period L is known and $N = L/M$. Sometimes, when considering the Zak transform generally and not a particular sample, we shall write $Z_M(g)$ or even $Z(g)$, if M is assumed to be known. We shall also sometimes refer to the Zak transform with block size M as the M-Zak transform.

Using the block DFT model for the Zak transform, the following characteristics are noted.

1. The Zak transform can be implemented as an incomplete FFT, and therefore, takes less time. The time needed to compute the FFT is proportional to $L \log L$. Therefore the time needed for the $N \times M$ Zak transform is proportional to $L \log(L/M)$

2. The Discrete Fourier Transform (DFT) is a particular case of the Zak transform with the block size M equal to 1.

3. The original signal can be obtained from its Zak transform by making the inverse DFT of the size N on the blocks:

$$g[m+iM] = \frac{1}{N}\sum_{n=0}^{N-1} Z_M(g)(n,m)e^{j\frac{2\pi}{N}ni}, 0 \le i < N, 0 \le m < M. \tag{2}$$

4. The Zak transform is a linear transformation that preserves the inner products (up to a constant factor), i.e., it is a linear isometry.

Now we shall consider the properties of the Zak transform upon which this paper is based.

Theorem 1 *Let $s[k]$, a periodic signal with period L, be the input to a filter $h[k]$ with periodic impulse response of the same period L. The output after decimation by M is*

$$c[l] = \sum_{k=0}^{L-1} h[k]s[Ml-k] = \sum_{k=0}^{L-1} h[Ml-k]s[k], \tag{3}$$

for $l = 0, 1 \ldots, N-1$, $NM = L$. Then

$$c[l] = \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{m=0}^{M-1} Z_M(h)(n,m)Z_M(s)(n,-m)\right)e^{j\frac{2\pi}{N}ln}. \tag{4}$$

That is, the output coefficients $c[l]$, are the inverse N-point FFT of the sum of the products of the samples of $N \times M$ Zak transforms of the filter impulse response and the signal reversed within each block.

See Appendix for the proof.

An analogous theorem for the continuous Zak transform was proved in [1]. The proof of the theorem is very similar to the proof of the theorem about circular convolution in the frequency domain of the DFT (see [4]) page 546). By the way, the circular convolution theorem of the DFT is a particular case of the above theorem when $M = 1$.

Suppose that the function $h[k]$ has compact support of length $L = MN$. Let us denote its time reversal $h^-[k] = h[-k]$. Consider the circular M-shifts of the function $h_l^- = h^-[k - Ml] = h[Ml - k]$, $l = 0, \ldots, N-1$, where all the arguments are taken $\bmod L$. Since we shall encounter this construction frequently, we provide an example of the circular shifts.

If $h = [1,2,3,4,5,6]$. Then $h^- = [6,5,4,3,2,1]$. There are three 2-shifts of h^-: $h_0^- = h^-$, $h_1^- = [4,3,2,1,6,5]$ and $h_2^- = [2,1,6,5,4,3]$.

Another interpretation of the circular shifts is obtained by extending the function periodically and considering its usual shift.

The following theorem states that given the coefficients $c[l]$ we can always find the projection of the signal upon the span of circular shifts of the function $\bar{h}^-$, the conjugate of h^-. Also the theorem gives a procedure for finding these coefficients.

Theorem 2 *Following the notation above one can find the M-Zak transform of the projection $s_{\bar{h}^-}[k]$ of the signal $s[k]$ upon the span of the functions $\{\bar{h}_l^-\}_{l=0}^{N-1}$ (where overbar denotes complex conjugate), by doing the following:*

1. Take the DFT of the convolution coefficients $c[l]$:

$$d[l] = \sum_{i=0}^{N-1} c[i] e^{-j\frac{2\pi}{N}il}. \tag{5}$$

2. Find the square of the norm of each block, for $l = 0, \ldots, N-1$, of the Zak transform of $h[k]$:

$$H_M[l] = \sum_{r=0}^{M-1} |Z_M(h)(l,r)|^2, \tag{6}$$

for $l = 0, 1, \ldots, N-1$.

3. To find the Zak transform of the projection, multiply each block of the complex conjugate of the Zak transform of the function $h[k]$ by the corresponding coefficient $d[l]$ divided by the square of the norm of each block, for those blocks, whose square norm is not zero, of the Zak transform of $h[k]$, treating blocks as vectors and coefficients as scalars:

$$Z_M(s_{\bar{h}^-})(l,m) = \begin{cases} \bar{Z}_M(h)(l,-m)\frac{d[l]}{H_M[l]} & \text{if } H_M[l] \neq 0 \\ 0 & \text{if } H_M[l] = 0, \end{cases}, \tag{7}$$

for $l = 0, 1, \ldots, N-1$, $m = 0, 1, \ldots, M-1$, $H_M[l] \neq 0$, *where the bar denotes complex conjugate.*

To obtain the projection itself one should take the inverse Zak transform of $Z_M(s_{\bar{h}^-})$*:*

$$s_{\bar{h}^-}[Mn+m] = \frac{1}{N}\sum_{l=0}^{N-1} Z_M(s_{\bar{h}^-})(l,m)e^{j\frac{2\pi}{N}ln}, \tag{8}$$

for $n = 0, 1, \ldots, N-1$, $m = 0, 1, \ldots, M-1$.

See Appendix for the proof.

The function h^- was considered instead of h for the purpose of implementing the above procedure as a convolution (filtering).

It is possible to shorten the above procedure when the coefficients $c[n]$ are not known. Instead of obtaining first the coefficients $c[n]$ and then $d[n]$, one can use (76) directly to obtain the coefficients $d[n]$.

The next theorem enables one to make a simple check whether the functional set $\{\bar{h}_l^-\}_{l=0}^{N-1}$ (or, equivalently, set $\{h_l[k] = h[k-lM]\}_{l=0}^{N-1}$) is linearly independent.

Theorem 3 *The set of L-periodic functions,* $L = MN$, $\{h_l\}_{l=0}^{N-1}$ *or equivalently the set* $\{\bar{h}_l^-\}_{l=0}^{N-1}$ *(subscript denoting the number of M-shifts) is linearly independent iff the square of the norm of each block,* $H_M[k]$ *as defined in (6), is not equal to zero for each* $k = 0, 1, \ldots, N-1$.

See Appendix for the proof.

3 Filter Banks

3.1 Coefficients of the Decomposition, Biorthogonal Filters and Best Reconstruction in Single Filter Case.

In the previous section it was shown how to find the projection $s_{\bar{h}^-}[k]$ of the signal $s[k]$ upon the span of the functions $\{\bar{h}_l^-\}_{l=0}^{N-1}$. Now the algorithm for finding the coefficients of the decomposition of $s_{\bar{h}^-}[k]$ upon the set $\{\bar{h}_l^-\}_{l=0}^{N-1}$ is considered.

Theorem 4 *In the representation of the* $s_{\bar{h}^-}$ *as a linear combination of functions from the set* $\{\bar{h}_l^-\}_{l=0}^{N-1}$*, the coefficients* $\tilde{c}[l]$ *in the expansion*

$$s_{\bar{h}^-}[k] = \sum_{l=0}^{N-1} \tilde{c}[l]\bar{h}_l^-[k]. \tag{9}$$

can be found (not necessarily uniquely) according to the following formula:

$$\tilde{c}[l] = \frac{1}{N} \sum_{n=0, H_M[n]\neq 0}^{N-1} \sum_{m=0}^{M-1} \frac{Z_M(h)(n,m)}{H_M[n]} Z_M(s)(n,-m) e^{j\frac{2\pi}{N}ln}. \tag{10}$$

See Appendix for the proof.

Theorem 5 *The coefficients $\tilde{c}[n]$ can be obtained by passing the signal through a filter whose impulse response $\tilde{h}[k]$, is given by*

$$\tilde{h}[m+iM] = \frac{1}{N} \sum_{n=0, H_M[n]\neq 0}^{N-1} \frac{Z_M(h)(n,m)}{H_M[n]} e^{j\frac{2\pi}{N}ni},$$

$$0 \le i < N, 0 \le m < M \tag{11}$$

which is the inverse M-Zak transform of $\frac{Z_M(h)(n,m)}{H_M[n]}$, with subsequent M-decimation. Notice that

$$Z_M(\tilde{h})(n,m) = \begin{cases} \frac{Z_M(h)(n,m)}{H_M[n]}. & \textit{if } H_M[n] \neq 0 \\ 0 & \textit{if } H_M[n] = 0 \end{cases} \tag{12}$$

See Appendix for the proof.

The filter with such impulse response is called biorthogonal (with respect to the original filter with impulse response $h[k]$). The reason for it is that the impulse resonse and its M shifts constitute a basis biortogonal to the one represented by $h[k]$ and its M-shifts in the subspace, spanned by the bases. In linear algebra biorthogonality of two bases spanning the same space means that the decomposition coefficients of an arbitrary vector on one of the bases can be found by taking the inner products with the vectors of the other basis.

It follows from equation (12) that the biorthogonality property is reciprocal: if a filter A is biorthogonal to a filter B, then the filter B is biorthogonal to the filter A.

Notice that also from (12) one deduces that

$$\sum_m Z_M(\tilde{h})(n,m)\bar{Z}_M(h)(n,m) = \frac{\sum_m Z_M(h)(n,m)\bar{Z}_M(h)(n,m)}{H_M[n]} = \frac{H_M[k]}{H_M[k]} = 1, \tag{13}$$

for every $n = 0, 1, \ldots, N$. In other words if we construct vectors (or one-dimensional matrices) $\boldsymbol{H}(n)$=$(Z_M(h)(n,0)$,$Z_M(h)(n,1)$,...,$Z_M(h)(n,M-1))$ and $\tilde{\boldsymbol{H}}(n)$=$(Z_M(\tilde{h})(n,0)$,$Z_M(\tilde{h})(n,1)$,..., $Z_M(\tilde{h})(n,M-1))$, then we can write the condition (13) as

$$\boldsymbol{H}(n)\tilde{\boldsymbol{H}}^*(n) = \boldsymbol{I}_1, \tag{14}$$

where $\tilde{\boldsymbol{H}}^*$ is the complex transpose of $\tilde{\boldsymbol{H}}$ and $\boldsymbol{I}_1$ is the 1×1 identity matrix.

Let us now associate the filtering-decimation and interpolation-filtering operations to the mathematical formulas. The signal $p[k]$ after filtering and M-decimation is already given according to the expression in (3), and the scheme is depicted in Figure 2. The M-interpolation-filtering scheme is shown in Figure 3. First the signal $p[k]$ is M-interpolated, i.e. M zeros are inserted between each of the two sequential elements of $p[k]$. The resulting (upsampled) signal $u[k]$ is passing through a filter $g[k]$. Assuming that everything is L-periodic in time and $L = NM$ the output signal $r[k]$ is then given by the following formula:

$$r[l] = \sum_{k=0}^{L-1} u[k]g[l-k] = \sum_{k=0}^{N-1} p[k]g[l-kM]. \tag{15}$$

Notice that the equation above states that the $r[k]$ is in the linear span of functional set $\{g_k[l] = g[l-kM]\}_{k=0}^{N-1}$, and $p[k]$ are the coefficients of the decomposition.

Suppose, now that we follow the scheme shown in the Figure 4. If we take $f[k] = h[k]$ and $g[k] = \bar{\tilde{h}}[-k]$, then according to Theorem 5 and the reciprocality of the biorthogonal bases, the signal $p[k]$ will represent the coefficients of the decomposition of the projection of the input signal upon the span of the set $\bar{\tilde{H}}^{-} = \{\bar{\tilde{h}}_k^{-}\}_{k=0}^{N-1}$, where $\bar{\tilde{h}}_k^{-}[l] = \bar{\tilde{h}}[kM-l]$. Then after the second filtering one obtains the projection $s_{\bar{\tilde{h}}^-}$ of the input signal s upon the span of $\bar{\tilde{H}}^{-}$ itself. Since $span(\bar{\tilde{H}}^{-}) = span(\bar{H}^-)$, where $\bar{H}^- = \{\bar{h}_k^-\}_{k=0}^{N-1}$, the output signal is also a projection $s_{\bar{h}^-}$ of the input signal upon $span(\bar{H}^-)$. The reconstruction therefore is exact for input signals belonging to $span(\bar{H}^-)$. Also the above reasoning shows that if we fix the output filter $g[k]$, the input filter $f[k]$ that produces at the output the best (in the mean square sense) approximation of the input is $f[k] = \bar{\tilde{g}}[-k]$. The inverse is not true: if one chooses to fix the input filter, then for different input signals different output filters will produce the best input approximations. Therefore we call the scheme in Figure 4 with $g[k] = \bar{\tilde{f}}[-k]$ the optimal single filter filtering-decimation-reconstruction scheme.

Introducing biorthogonal filters, one can perform both reconstruction and decomposition in the time domain, which, for short filters, is faster than going to the Zak transform domain.

3.2 Optimal Filter Banks, Orthogonalization and Perfect Reconstruction

In this subsection the filter banks consist of M parallel analysis and synthesis arms, as depicted in Figure 5. The chosen approach is orthogonalization of filters in the Zak domain. We start with a short lemma.

Lemma 1 *The M-Zak transform $Z_M(w_p)$ of a function*

$$w_p[k] = Z_M(h)(p,k) = \sum_{l=0}^{N-1} h[k+lM]e^{-j\frac{2\pi}{N}pl}, \tag{16}$$

is given by the following formula [1]*:*

$$Z_M(w_p)(n,m) = Z_M(h)(n,m)N\delta[n-p] = \begin{cases} NZ_M(h)(p,m) & \text{if } n=p \\ 0 & \text{if } n \neq p, \end{cases} \tag{17}$$

for $n = 0, 1, \ldots, N-1$.

See Appendix for the proof.

The consequences of Lemma 1 are that (17) implies that

$$Z_M(h)(p,m) = \frac{1}{N}\sum_{n=0}^{N-1} Z_M(w_n)(p,m). \tag{18}$$

and that (16) implies that

$$w_p \in span(\{h_l\}_{l=0}^{N-1}), \tag{19}$$

for $p = 0, 1, \ldots, N-1$.

Recall that by the nth block of the Zak transform $Z_M(h)(n,m)$ of an L-periodic function h, we mean the set of values $Z_M(h)(n,m)$ for $m = 0, 1, \ldots, M-1$. Actually, since this set of values determines the values of the Zak transform for any integer m, we can call the nth block of the Zak transform the set of numbers $Z_M(h)(n,m)$ for fixed n and any integer m.

One of the direct consequences of the lemma above is the following theorem

Theorem 6 *Consider K L-periodic functions $h^0, h^1, \ldots, h^{K-1}$ and their M-Zak transforms $Z_M(h^0), Z_M(h^1), \ldots, Z_M(h^{K-1})$. Consider also the function f whose Zak transform M-blocks are linear combinations of the M-blocks of*
$Z_M(h^0), Z_M(h^1), \ldots, Z_M(h^{K-1})$:

$$Z_M(f)(n,m) = \sum_{s=0}^{K-1} a_{s,n} Z_M(h^s)(n,m), \tag{20}$$

where $a_{s,n}$ are any complex numbers, that may vary in s and in n. Then the function f will belong to the span of the functions $h^0, h^1, \ldots, h^{K-1}$ and their M-shifts, $span(\{h^s_n\}_{n=0,s=0}^{N-1,S-1})$, where the subscript denotes the number of M-shifts. The converse is also true: the Zak transform of every function $f \in span(\{h^s_n\}_{n=0,s=0}^{N-1,S-1})$ can be represented by (20).

[1] The subscript of the function w will always denote a parameter of the function w, such as the parameter p in w_p above, while the subscript of any other function will continue to denote the number of M-shifts.

See Appendix for the proof.

Theorem 7 *Consider functional sets* $\{h_l^0\}_{l=0}^{N-1}$ *and* $\{h_l^1\}_{l=0}^{N-1}$*, where the subindex denotes the number of* M*-shift, for any two arbitrary* L*-periodic functions* h^0 *and* h^1*. Then every function of one set is orthogonal to every function of the other set iff for every* n

$$\sum_m Z_M(h^0)(n,m)\bar{Z}_M(h^1)(n,m) = 0. \tag{21}$$

See Appendix for the proof.

To find the optimal biorthogonal solution for the filter banks, when the number of filters K is, in general, less than or equal to the decimation number M, it seems reasonable to orthogonalize the M-shifts of the impulse responses of the filters using Zak transform technique, to obtain the coefficients of the best approximation (projection) of the input signal upon the mutually orthogonal sets of functions and then take the linear transformation of the coefficients of the decomposition to obtain the coefficients of the decomposition upon the original basis. The following theorems formalize the above reasoning.

Theorem 8 *Consider* K L*-periodic functions* $h^0, h^1, \ldots, h^{K-1}$ *and their* M*-shifts* $\{h_l^0\}_{l=0}^{N-1}, \{h_l^1\}_{l=0}^{N-1}, \ldots, \{h_l^{K-1}\}_{l=0}^{N-1}$*. Consider the matrix of the Zak transforms of the functions:*

$$\boldsymbol{H}[n] = \begin{pmatrix} Z_M(h^0)(n,0) & Z_M(h^0)(n,1) & ,\ldots, & Z_M(h^0)(n,M-1) \\ Z_M(h^1)(n,0) & Z_M(h^1)(n,1) & ,\ldots, & Z_M(h^1)(n,M-1) \\ \cdots & \cdots & \cdots & \cdots \\ Z_M(h^{K-1})(n,0) & Z_M(h^{K-1})(n,1) & ,\ldots, & Z_M(h^{K-1})(n,M-1) \end{pmatrix}$$

In the language of filter banks, the matrix $\boldsymbol{H}[n]$ *is called the polyphase matrix of the filter bank whose filters have impulse response* $h^0, h^1, \ldots, h^{K-1}$*. Suppose that there is a non-singular* $K \times K$ *matrix* $\boldsymbol{O}[n]$ *that orthogonalizes the rows of* $\boldsymbol{H}[n]$ *for every* n*, i.e. the matrix*

$$\boldsymbol{V}[n] = \boldsymbol{O}[n]\boldsymbol{H}[n] \tag{22}$$

has orthogonal row vectors for every n*. The rows of the matrix* $\boldsymbol{V}[n]$ *can be considered to be the Zak transform blocks of some functions* $v^0, v^1, \ldots, v^{K-1}$*, so that*

$$\boldsymbol{V}[n] = \begin{pmatrix} Z_M(v^0)(n,0) & Z_M(v^0)(n,1) & ,\ldots, & Z_M(v^0)(n,M-1) \\ Z_M(v^1)(n,0) & Z_M(v^1)(n,1) & ,\ldots, & Z_M(v^1)(n,M-1) \\ \cdots & \cdots & \cdots & \cdots \\ Z_M(v^{K-1})(n,0) & Z_M(v^{K-1})(n,1) & ,\ldots, & Z_M(v^{K-1})(n,M-1) \end{pmatrix} \tag{23}$$

Then the following is true:

1. *The functions* $\{v^q\}_{q=0}^{N-1}$ *and their* M*-shifts span the same subspace as the functions* $\{h^q\}_{q=0}^{N-1}$ *and their* M*-shifts.*

2. *To obtain the Zak Transform of the projection* $s_{\bar{h}^-}$ *of a signal* s *upon the span of the functions* $\{\bar{h}_l^{q-}\}_{l=0,q=0}^{N-1,K-1}$ *(functions* $\{\bar{h}^{q-}\}_{q=0}^{K-1}$ *and their* M*-shifts) one can use the following formula:*

$$Z_M(s_{\bar{h}^-})(n,k) = \sum_{q=0}^{K-1} \bar{Z}_M(v^q)(n,-k)\tilde{w}^q[n], \tag{24}$$

where

$$\tilde{w}^q[n] = \begin{cases} \frac{\sum_{m=0}^{M-1} Z_M(v^q)(n,m)Z_M(s)(n,-m)}{V_M^q[n]}. & \text{if } V_M^q[n] \neq 0 \\ 0 & \text{if } V_M^q[n] = 0 \end{cases} \tag{25}$$

and

$$V_M^q[n] = \sum_{m=0}^{M-1} |Z_M(v^q)(n,m)|^2. \tag{26}$$

See Appendix for the proof.

Now we will present a theorem on biorthogonal filter banks.

Theorem 9 *Consider the set of kernels and their shifts* $\{\bar{h}_l^{q-}\}_{l=0,q=0}^{N-1,K-1}$. *To find the coefficients* $\{\tilde{c}^q[l]\}_{l=0,q=0}^{N-1,K-1}$ *of the decomposition of the projection of a signal* s *upon the span of the set, one can first find* $\{\tilde{d}^q[n]\}_{l=0,q=0}^{N-1,K-1}$, *which are the DFT of the coefficients* $\tilde{c}^q[l]$, *the following approach may be used: represent the coefficients* $\tilde{d}^q[n]$ *as a* K*-dimensional vector,* $\tilde{\boldsymbol{D}}[n] = (\tilde{d}^0[n], \tilde{d}^1[n], \ldots, \tilde{d}^{K-1}[n])$*; construct a column vector* $\boldsymbol{S}^-[n] = (Z_M(s)(n,0),\ Z_M(s)(n,-1), \ldots,\ Z_M(s)(n,-M+1))^T$*; and then using the orthogonalization of the Zak Transform matrix as in the previous theorem, one determines the coefficients* $\tilde{d}^q[n]$ *using the formula,*

$$\tilde{\boldsymbol{D}}[n] = \tilde{\boldsymbol{W}}[n]\bar{\boldsymbol{O}}[n], \tag{27}$$

where

$$\tilde{\boldsymbol{W}}[n] = [\tilde{w}^0[n], \tilde{w}^1[n], \ldots, \tilde{w}^{K-1}[n]],$$

and $\tilde{w}^q[n]$ *are given by (25). Another way to find the coefficients is through the formula:*

$$(\tilde{\boldsymbol{D}}[n])^T = (\boldsymbol{H}[n]\boldsymbol{H}^*[n])^{-1}\boldsymbol{H}[n](\boldsymbol{S}^-[n]), \tag{28}$$

where $\boldsymbol{H}[n]$ *is the same as in the previous theorem. To find the coefficients* $\tilde{c}^q[l]$ *one should take the inverse DFT of* $\tilde{d}^q[n]$*:*

$$\tilde{c}^q[l] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{d}^q[n]e^{j\frac{2\pi}{N}nl}. \tag{29}$$

See Appendix for the proof.

A few remarks should be made. First we purposely did not require $(\boldsymbol{H}[n]\boldsymbol{H}^*[n])$ to be invertible, i.e. rows of $\boldsymbol{H}[n]$ to be independent. Indeed suppose (without loss of generality) that the first Q of the K rows are independent. Then one can consider a $Q \times M$ matrix instead of H, with only the first Q rows retained, and obtain the best solution using the approaches developed in the theorem above. Also it can be shown that in such a case one can obtain the minimal energy solution by taking the pseudo inverse of $(\boldsymbol{H}[n]\boldsymbol{H}^*[n])$.

The above theorem shows two ways of finding the coefficients $\tilde{d}^q[n]$: through the formula (27) or through the formula (28). It is obvious that it is shorter to find the coefficients through the formula (27). However the formula (28) gives the expression for the biorthogonal filter bank and a way of finding the coefficients $\tilde{c}^q[l]$ in the time domain as is shown by the following theorem, which is an analogue of Theorem 5 for filter banks.

Theorem 10 *With conditions as above, to obtain the coefficients $\tilde{c}^q[l]$ one can pass the signal $s[k]$ through the filter bank, with polyphase matrix*

$$\tilde{\boldsymbol{H}}[n] = (\boldsymbol{H}[n]\boldsymbol{H}^*[n])^{-1}\boldsymbol{H}[n]. \tag{30}$$

See Appendix for the proof.

Therefore, following the discussion in case of a single filter, given a filter bank, shown in Figure 5, if we take $f^q[k] = h^q[k]$ and $g^q[k] = \tilde{\bar{h}}^q[k]$ then, at the output, we will obtain the projection $s_{\bar{h}^-}$ of the signal s upon the span of $\{\bar{h}_l^{q-}\}_{l=0,q=0}^{N-1,K-1}$. Perfect reconstruction is achieved if and only if the matrix $\boldsymbol{H}[n]$ does not have any element which is zero for every n. Furthermore, one can show that the oversampling case $(K > M)$ can also be handled with this technique.

3.3 Orthogonalization Algorithm

Until now nothing was said about how to find the orthogonalization matrix $\boldsymbol{O}[n]$, which one needs to find $\tilde{\boldsymbol{D}}[n]$ in (27). To find the biorthogonal filter, one would also need $(\boldsymbol{H}[n]\boldsymbol{H}^*[n])^{-1}$, which can be found in the following way through (22) :

$$(\boldsymbol{H}[n]\boldsymbol{H}^*[n])^{-1} = ((\boldsymbol{O}[n])^{-1}\boldsymbol{V}[n]((\boldsymbol{O}[n])^{-1}\boldsymbol{V}[n])^*)^{-1} = \boldsymbol{O}^*[n](\boldsymbol{V}[n]\boldsymbol{V}^*[n])^{-1}\boldsymbol{O}[n]. \tag{31}$$

Since $(\boldsymbol{V}[n]\boldsymbol{V}^*[n])$ is a diagonal matrix, if $\boldsymbol{O}[n]$ is known, the inversion is straightforward.

The matrix $\boldsymbol{O}[n]$ may be found through the Gram-Schmidt procedure, first letting $Z_M(v^0)(n,m) = Z_M(h^0)(n,m)$ for all n and m. To find $Z_M(v^q)(n,m)$, assuming that $Z_M(v^p)(n,m)$ has already been found for $p < q$, let

$$Z_M(v^q)(n,m) = Z_M(h^q)(n,m) - \sum_{p=0}^{q-1} \frac{Z_M(v^p)(n,m)(\sum_{k=0}^{M-1} Z_M(h^q)(n,k)\bar{Z}_M(v^p)(n,k))}{V^q[n]}, \quad (32)$$

where $V^q[n]$ is given in (26). Denoting $\alpha^{p,q}[n] = \sum_{k=0}^{M-1} Z_M(h^q)(n,k)\bar{Z}_M(v^p)(n,k)$, obtain

$$Z_M(v^q)(n,m) = Z_M(h^q)(n,m) - \frac{\sum_{p=0}^{q-1} Z_M(v^p)(n,m)\alpha^{p,q}[n]}{V^q[n]}. \quad (33)$$

Proceeding in the same way until $q = K$ we obtain the complete orthogonalization. On the qth stage the previously obtained matrix is being multiplied by

$$\boldsymbol{O}^q[n] = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\alpha^{0,q}[n] & -\alpha^{1,q}[n] & \dots & -\alpha^{q-1,q}[n] & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{pmatrix}. \quad (34)$$

The matrix $\boldsymbol{O}[n]$ is, correspondingly, the product of these matrices:

$$\boldsymbol{O}[n] = \boldsymbol{O}^{K-1}[n]\boldsymbol{O}^{K-2}[n]\dots\boldsymbol{O}^1[n]. \quad (35)$$

Notice that the resulting sets $\{v_l^0\}_{l=0}^{N-1}, \{v_l^1\}_{l=0}^{N-1}, \dots, \{v_l^{K-1}\}_{l=0}^{N-1}$ are orthogonal to one another (each functions of one of the sets is orthogonal to every function of another). Therefore the procedure above can be considered to be the fast orthogonalization algorithm of the functional sets $\{h_l^0\}_{l=0}^{N-1}, \{h_l^1\}_{l=0}^{N-1}, \dots, \{h_l^{K-1}\}_{l=0}^{N-1}$ in the Zak Transform domain. In the time domain orthogonalization would require much greater computational complexity.

4 Non-Periodic Case

4.1 The Non-Periodic Formulation

All of the above definitions and theorems can be formulated for the non-periodic case as well. Below, we give the definition of the Zak Transform for the non-periodic case.

Definition 2 *Let $g[k]$ be function from l_2 (the space of all square summable sequences). The discrete M Zak transform of the function $g[k]$ is*

$$\begin{aligned} Z_M(g)(\omega, m) &= \sum_{i=-\infty}^{\infty} g[m+iM]e^{-j2\pi\omega i}, \\ &\quad -\pi \leq \omega \leq \pi, 0 \leq m < M-1. \end{aligned} \tag{36}$$

Even though the theory is fully applicable for non-periodic filters and signals, in practice, we cannot conduct the filtering in the frequency domain in this case, since the filters would have a continuous frequency response. The theory however, leads to some very interesting time (space) domain theoretical and practical filtering results, see [21].

4.2 Finding the Biorthogonal Non-Periodic Filters on Practice

Suppose we have a non-periodic finite support filter h. We consider the M-shifts of h and want to find the biorthogonal filter $\tilde{h}$. One way to do it is to consider a periodic filter that is a periodic extension of a part of h, that includes all of its support and the padding of zeros around the support. The more we increase the period of the filter, by padding more zeros around the support, the closer will be the obtained biorthogonal filter to the biorthogonal filter of the non-periodic case. This property is used in the section 4.4 to find the biorthogonal filter pairs.

4.3 Mirror Filters and the Wavelet Transforms

Wavelet transform for the discrete signals was treated in detail in [13], [19], [17], [9], [8] and other articles. In short we have two filters: the low-pass h and the high-pass g. To perform the wavelet transform, we filter a signal s, with those filters and decimate the filtering results by 2. Then we apply the same procedure to the low-band recursively until the required resolution level is reached. The wavelet decomposition is very suitable for image and signal compression as was shown e.g. in [18] and [20].

The main problem of the wavelet transform is the reconstruction of the original signal. Given filters h and g one needs to find the reconstruction filters h^r and g^r that would guarantee perfect reconstruction for the filter bank as the one in Figure 5 with $M = 2$, $f^0 = h$, $f^1 = g$, $g^0 = h^r$, and $g^1 = g^r$.

As shown in [13] and [19], in case of orthogonal filters, the reconstruction filters can be the same as the decomposition filters. However the orthogonality condition puts too much of a restriction on the filter choice, which leads to filters whose compression potential is not very high. The biorthogonal filters considered in [9], [8] and [20] are better suited for compression.

Given a filter h let us call the filter $h^M[k] = (-1)^k g[1-k]$, its mirror filter. The traditional way of performing the biorthogonal wavelet transform is to take the reconstruction filters $h^r = g^M$ and $g^r = h^M$ to be the mirror filters of the decomposition filters, which leads to some condition on the choice of the low and high pass filters.

We propose a different approach. The 2-shifts of the mirror filters are orthogonal to one another (it can be seen by considering their inner products). To use the theory developed above, any filter h with a non-zero sum of of the samples may serve as the low-pass and its mirror filter h^M as the high-pass wavelet filter pair. Then reconstruction may be accomplished using the biorthogonal filters. Moreover it is easy to show that the filter biorthogonal to the mirror filter is the mirror filter of the biorthogonal filter, so the reconstruction pair is $\tilde{h}$ and $\tilde{h}^M$. The 2-shifts of the decomposition and reconstruction filters are thus spanning two mutually orthogonal subspaces of l^2. Notice that because of (12) and Theorem 6, the M-shifts of mutually biorthogonal filters have the same span. Thus the 2-shifts of $\tilde{h}$ span the same subspace of l^2 as the 2-shifts of h, orthogonal to the spans of the 2-shifts of h^M and $\tilde{h}^M$. That means that $\tilde{h}$ can be used for decomposition together with h^M, while h and $\tilde{h}^M$ can be used as the reconstruction filters. Coding and reconstruction experiments, described below, show that for short h, wavelet transforms with such filter pairs were producing the best results.

4.4 Examples of Biorthogonal Filters and Compression Results

We used the technique described above to perform the 2-D separable wavelet transform on an image, compress the wavelet coefficients, using the SPIHT algorithm without entropy coding (see [18]), and restore the image.

In order to find filters that produce the best compression results, we conducted the experiments with filters h of support sizes from 3 to 7. As stated in the previous subsection, we used the biorthogonal filter $\tilde{h}$ as the low-pass decomposition filter, g, the mirror filter of h, as the high-pass decomposition filter, h itself as the low-pass reconstruction filter and the biorthogonal filter of the mirror filter of h, $\tilde{g}$ as the high-pass reconstruction filter. The only restriction on the low pass filter impulse responses h was that they were chosen to be linear phase. For every support size we chose the limits within which the filter tap values can vary (which were quite large) and the step size for changing the tap values. We obtained the compression results for all of such filters with one fixed compression rate (0.25 bpp). We picked out the filters that produce the best compression results under this rate for each of the support sizes. In [21] we show that the performance of the filters is steadily good for all of the rates. For example, our 7 tap filter beats the well-known 9/7 filter, see [20], for all of the compression rates from 0.15 to 0.5 bpp on the Lena and Barbara 512×512 luminance images, with Barbara showing the larger improvement.

The biorthogonal filters $\tilde{h}$ and $\tilde{g}$ were obtained using the procedure described in the previous subsection. First the original filters h and g were padded with zeros symmetrically on both sides to bring the total number of samples to 128. Then the filters biorthogonal to them were found using the formular (11) of the theorem 5. The segments 48-th to 80-th pixels of the biorthogonal filters impulse responses, which included virtually all of their supports were plotted on the same plots as the same segments of the original filters The 48-th sample thus corresponds to the 1-st sample on the plot. The center (peak) of the filters falls on the 16-th pixel of the plots. On the plots the original filters are normalized, i.e. they are multiplied by a factor to bring their sum to $\sqrt{2}$. The plots are shown in Figures 6, 7 and 8. One can see that the biorthogonal filter supports are quite large, which causes increase in computational complexity. However, as stated in [21] these filters can be implemented very fast as recursive filters. The best filters and peak-to-peak signal to noise ratio (PSNR) coding results for the rate 0.25 bpp using the Lena image, are compiled in the Table 4.4. The PSNR is defined by

$$PSNR = 10 \log_{10}(\frac{255^2}{MSE}) dB. \tag{37}$$

The result for the 7 tap filter is 0.09dB better than that for the 9/7 filter bank (9 tap low pass and 7 tap high pass decomposition filters with 7 tap low pass and 9 tap high pass reconstruction filters, see [20]) which is one of the best among the existing filters.

5 Multiresolutional Properties of the Zak transform

In this section are considered properties of the Zak transform that will be applied in the derivation of the fast wavelet decomposition. First let us consider a property that will allow us to obtain the Zak transform of a signal with a smaller block size from one with a bigger block size (finer one from a coarser one).

Filter Comparison Results	
Filter h	PEAK-TO-PEAK SNR (dB)
9/7 Filter	33.69
[1 2 1]	33.09
[-1 2 10 10 2 -1]	33.59
[-1 -0.5 6 11 6 -0.5 -1]	33.78

Table 1: Comparison of performances of filters in compression of 512×512 Lena luminance image at 0.25 bpp.

Theorem 11 *Suppose that M_1 is a divisor of M, i.e., $M_1 = M/p$. To obtain the Zak transform with the block size M_1 from the one with the block size M, one can use the following equation:*

$$Z(s)(\frac{n}{Np}, \frac{m}{M_1}) = \sum_{l=0}^{p-1} Z(s)(\frac{n_{res}}{N}, \frac{m + lM_1}{M})e^{-j\frac{2\pi}{Np}ln}, \tag{38}$$

where n_{res} is the remainder of n after division by N (remember that n varies from 0 to $Np-1$). The notations here follow that of [2], since the number of blocks is changed in the formula.

The theorem can be proved by direct substitution of the expressions for the Zak transforms. Notice that if we chose $p = 2$ and apply the theorem recursively, finding first $Z_{L/2}(s)$ from the sequence s which can be assumed to be $Z_L(s)$, then $Z_{L/4}(s)$ from $Z_{L/2}(s)$ and so on, obtaining sequence $Z_{L/2^n}(s)$ from $Z_{L/2^{n-1}}(s)$, until we obtain $Z_1(s)$. This is just an implementation of the decimation-in-frequency FFT algorithm (see [4]). That means that by executing this FFT algorithm, we shall obtain Zak transforms of the sequence on each stage, with the size of the block assuming the values $L/2, L/4, \ldots, L/2^n, \ldots 1$.

There is also an inverse procedure, which will be useful later in the paper, that allows to compute a block size M Zak transform from a block size M_1 Zak transform when M is an integer multiple of M_1. The theorem below formalizes this procedure.

Theorem 12 *Suppose that $M = M_1 p$, where p is an integer, then*

$$Z_M(s)(n, m + lM_1) = \frac{1}{p} \sum_{q=0}^{p-1} Z_{M_1}(n + qN, m)e^{j\frac{2\pi}{Np}(n+qN)l}, \tag{39}$$

for $n = 0, 1, \ldots, N-1$, $m = 0, 1, \ldots, M_1 - 1$ and $l = 0, 1, \ldots, p-1$.

The theorem can again be proved by direct substitution of the expressions for the Zak transforms into (39).

The following theorem will also be useful in the next subsection.

Theorem 13 *Suppose that there is an L-periodic function $h[k]$, from which one constructs an $L/2$-periodic function $h^1[k] = h[k] + h[k + L/2]$. Then for every block size $M = 1, 2, \ldots L/2$, the Zak transform of h^1 is obtained from the Zak transform with the same block size of the function h by erasing every second block: $Z_M(h^1)(l, m) = Z_M(h)(2l, m)$. Generalizing, one can see that for the recursive sequence*

$$h^n[k] = h^{n-1}[k] + h^{n-1}[k + L/2^n] = \sum_{l=0}^{2^n} h[k + lL/2^n], \tag{40}$$

the following formula will hold: $Z_M(h^n)(l, m) = Z_M(h)(2^n l, m)$, for $M = 1, 2, \ldots L/2^n$.

The theorem can again be proved by direct substitution.

6 Fast Wavelet Transform

6.1 Background on the Wavelet Transform and Filter Banks.

In this subsection we will describe briefly the wavelet transform algorithm in the discrete signal domain without reference to its meaning in the continuous domain. Suppose two orthogonal filters with orthogonal impulse responses $h[k]$ and $g[k]$ are given (here orthogonality means that the 2-shifts of h and g span two mutually orthogonal subspaces of the space of all discrete functions). Then we just filter the signal $s[k]$ using these filters and decimate the obtained signals by the factor 2, obtaining signals s_h and s_g. To obtain the wavelet decomposition coefficients corresponding to the next resolution, the same procedure is applied to the signal s_h, and the signals s_{hh} and s_{hg} are obtained, and so on, getting coefficients corresponding to coarser and coarser resolution levels. For the reconstruction of $s[k]$ one must use the biorthogonal filters $\tilde{h}$ and $\tilde{g}$. To obtain, say, s_h from the signals s_{hh} and s_{hg}, first we interpolate by inserting zeros between the successive samples of the signals and then apply the interpolated s_{hh} to the filter $\tilde{h}$ and the interpolated s_{hg} to the filter $\tilde{g}$ and sum the results. The operations are shown in Figure 5 for $M = 2$, $f^0 = h$, $f^1 = g$, $g^0 = \tilde{h}$, and $g^1 = \tilde{g}$.

In the case where the same filter bank is used both for reconstruction and analysis, the filters should satisfy the QMF relationship, e.g. (see [13]). However, for the circular case, our methods provide a simple way to obtain the biorthogonal filters, so that the QMF property is no longer necessary. Moreover, it allows us to perform the full algorithm in the the Zak Transform domain. Also, there are no limitations due to the length of the impulse response of the filter, since the computational complexity depends on the signal length, not the filter length.

6.2 Algorithm for Fast Circular Wavelet Transform

First of all, it is necessary to say that the algorithm presented below is, as far as the authors can judge, in many aspects the same as the one developed by Vetterli ([8] and [16]). However since it never was described fully in English, we will present it here in detail. Also in certain aspects our case is more general, since from the theory of the Zak transform it will be easy to see how to make a generalization for M filters on the same octave instead of 2. Also we are stipulating the circular case, while Vetterli was stipulating the non-periodic one. Also the algorithm here implicitly includes finding the biorthogonal filters.

Let us assume that we have two mutually orthogonal filters with impulse responses $h[k]$ and $g[k]$. Suppose that the impulse responses of these filters are periodic with period L. Suppose also that we want to find J scales of the wavelet transform of an L-periodic signal $s[k]$ using these filters, where $J < \log_2 L$.

Let us take a closer look at the filtering process. At the first stage the aim is to obtain the convolutions of the signal s with impulse responses $h^0[k] = h[k]$ and $g^0[k] = g[k]$ and to decimate them, obtaining coefficients

$$c_h^1[k_1] = \sum_{k=0}^{L-1} s[k] h^0[2k_1 - k], \tag{41}$$

and

$$c_g^1[k_1] = \sum_{k=0}^{L-1} s[k] g^0[2k_1 - k], \tag{42}$$

for $k_1 = 0, 1 \ldots, \frac{L}{2} - 1$. On the second stage we apply these filters to the coefficients $c_h^1[k_1]$. Notice that since c_h^1 is $\frac{L}{2}$-periodic and functions $h[k]$ and $g[k]$ are L-periodic, two periods of $c_h[k_1]$ coefficients should be considered. The following sequences are obtained:

$$c_h^2[k_2] = \sum_{k_1=0}^{L-1} c_h^1[k_1] h^0[2k_2 - k_1] = \sum_{k_1=0}^{L/2-1} c_h^1[k_1](h^0[2k_2 - k_1] + h^0[2k_2 - k_1 - L/2]) \tag{43}$$

and

$$c_g^2[k_2] = \sum_{k_1=0}^{L-1} c_h^1[k_1] g^0[2k_2 - k_1] = \sum_{k_1=0}^{L/2-1} c_h^1[k_1](g^0[2k_2 - k_1] + g^0[2k_2 - k_1 - L/2]), \tag{44}$$

for $k_2 = 0, 1, \ldots, L/4 - 1$. Denoting $h^1[k] = h^0[k] + h^0[k + L/2]$ and $g^1[k] = g^0[k] + g^0[k + L/2]$, one obtains

$$c_h^2[k_2] = \sum_{k_1=0}^{L/2-1} c_h^1[k_1] h^1[2k_2 - k_1] \tag{45}$$

and

$$c_g^2[k_2] = \sum_{k_1=0}^{L/2-1} c_h^1[k_1] g^1[2k_2 - k_1]. \tag{46}$$

The procedure of obtaining the impulse responses of the next stage is the same as the one described in [11]. In general, coefficients corresponding to the nth level of resolution are obtained through the following formulas:

$$c_h^n[k_n] = \sum_{k_{n-1}=0}^{L/2^{n-1}-1} c_h^{n-1}[k_{n-1}] h^{n-1}[2k_n - k_{n-1}] \tag{47}$$

and

$$c_g^n[k_n] = \sum_{k_{n-1}=0}^{L/2^{n-1}-1} c_h^{n-1}[k_{n-1}] g^{n-1}[2k_n - k_{n-1}], \tag{48}$$

for $k_n = 0, 1, \ldots, L/2^n - 1$. In the formulas above

$$h^n[k] = h^{n-1}[k] + h^{n-1}[k + L/2^n] = \sum_{l=0}^{2^n-1} h^0[k + lL/2^n], \tag{49}$$

and

$$g^n[k] = g^{n-1}[k] + g^{n-1}[k + L/2^n] = \sum_{l=0}^{2^n-1} g^0[k + lL/2^n]. \tag{50}$$

Notice the analogy with the sequences of Theorem 13.

We continue this procedure until $n = J$, where J is the number of scales. On each stage, except for the last one, we are retaining only coefficients c_g^n, as the coefficients of the wavelet transform, while the coefficients c_h^n are being used for the next step of the decomposition. Only in the last stage are the coefficients c_h^J retained for subsequent reconstruction.

The fast algorithm for the wavelet decomposition whose diagram is shown on Figure 9, is described below. It is assumed that $Z_2(h^0)$ and $Z_2(g^0)$ are known and stored in memory.

1. Set $n = 1$, $c_h^0[k] = s[k]$.
2. Obtain $Z_2(c_h^0)(k_n, m) = \sum_{i=0}^{J-1} s[m + 2i] e^{-j\frac{2\pi}{L/2} i k_n}$.

3. Using the procedure of the Theorem 1, find the coefficients $d_h^n[k_n]$ and $d_g^n[k_n]$ using the following formulas:

$$d_h^n[k_n] = \sum_{m=0}^{1} Z_2(c_h^{n-1})(k_n, m) Z_2(h^{n-1})(k_n, -m) \tag{51}$$

and

$$d_g^n[k_n] = \sum_{m=0}^{1} Z_2(c_h^{n-1})(k_n, m) Z_2(g^{n-1})(k_n, -m), \tag{52}$$

for $k_n = 0, 1, \dots, L/2^n - 1$. Use of Theorem 13 and equations (49) and (50), reveals that

$$Z_2(h^{n-1})(l, m) = Z_2(h^0)(2^{n-1}l, m) \tag{53}$$

and

$$Z_2(g^{n-1})(l, m) = Z_2(g^0)(2^{n-1}l, m), \tag{54}$$

so that (51) and (52) can be rewritten as

$$d_h^n[k_n] = \sum_{m=0}^{1} Z_2(c_h^{n-1})(k_n, m) Z_2(h^0)(2^{n-1}k_n, -m) \tag{55}$$

and

$$d_g^n[k_n] = \sum_{m=0}^{1} Z_2(c_h^{n-1})(k_n, m) Z_2(g^0)(2^{n-1}k_n, -m), \tag{56}$$

for $k_n = 0, 1, \dots, L/2^n - 1$. The modified formulas are more convenient to use in the algorithm.

According to Theorem 1, the coefficients d_h^n and d_g^n are the DFTs of c_h^n and c_g^n.

4. Obtain coefficients c_g^n as the inverse DFT of the sequence d_g^n:

$$c_g^n[k_n] = \frac{2^n}{L} \sum_{l=0}^{L/2^n - 1} d_g^n[l] e^{j \frac{2\pi}{L/2^n} l k_n}, \tag{57}$$

for $k_n = 0, 1, \dots, L/2^n - 1$.

5. If $n = J$,

obtain coefficients c_h^n as the inverse DFT of the sequence d_h^n:

$$c_h^n[k_n] = \frac{2^n}{L} \sum_{l=0}^{L/2^n - 1} d_h^n[l] e^{j \frac{2\pi}{L/2^n} l k_n}, \tag{58}$$

for $k_n = 0, 1, \dots, L/2^n - 1$.

Stop.

If $n \neq J$,

go to step 6.

6. Since d_h^n is the DFT of c_h^n, one can write:

$$d_h^n[k_n] = Z_1(c_h^n)(k_n, 0). \tag{59}$$

Using the procedure developed in theorem 12, one can find $Z_2(c_h^n)$ according to the following formula:

$$\begin{aligned} Z_2(c_h^n)(k_{n+1}, l) &= \frac{1}{2}(Z_1(c_h^n)(k_{n+1}, 0)e^{j\frac{2\pi}{L/2^n}k_{n+1}l} + \\ &\quad Z_1(c_h^n)(k_{n+1} + L/2^{n+1}, 0)e^{j\frac{2\pi}{L/2^n}(k_{n+1}+L/2^{n+1})l}) \\ &= \frac{1}{2}(d_h^n[k_{n+1}] + (-1)^l d_h^n[k_{n+1} + L/2^{n+1}])e^{j\frac{2\pi}{L/2^n}k_{n+1}l}. \end{aligned} \tag{60}$$

7. Set $n = n + 1$. Go to step 3.

Now we shall consider the reconstruction algorithm, which is based on Theorem 2.

The diagram of the reconstruction algorithm is shown on the Figure 10.

First, as in Theorem 2 with $M = 2$, we find

$$H_2[l] = \sum_{r=0}^{1} |Z_2(h^0)(l, r)|^2, \tag{61}$$

and

$$G_2[l] = \sum_{r=0}^{1} |Z_2(g^0)(l, r)|^2, \tag{62}$$

for $l = 0, 1, \ldots, L/2 - 1$. Then, assuming that $H_2[l]$ and $H_2[l]$ are not equal to zero for every l (otherwise the complete reconstruction would not be possible), we proceed to derive the Zak Transform of the biorthogonal filters $\tilde{h}$ and $\tilde{g}$. Initially calculate

$$Z_2(\tilde{h^0})(l, r) = \frac{Z_2(h^0)(l, r)}{H_2[l]} \tag{63}$$

and

$$Z_2(\tilde{g^0})(l, r) = \frac{Z_2(g^0)(l, r)}{H_2[l]}. \tag{64}$$

Then the reconstruction goes according to the following algorithm:

1. Assuming that there are J levels of resolution, find $d_h^J[k_J]$ and $d_g^J[k_J]$ as DFTs of c_h^J and c_g^J correspondingly, for $k_J = 0, 1, \ldots, L/2^J - 1$.

2. Set $n = J$.

3. Find $Z_2(c_h^{n-1})$ according to the formula:

$$Z_2(c_h^{n-1})(k_n, m) = Z_2(\tilde{h^n})(k_n, -m)d_h^n[k_n] + Z_2(\tilde{g^n})(k_n, -m)d_g^n[k_n], \tag{65}$$

for $k_n = 0, 1, \ldots, L/2^n - 1$ and $m = 0, 1$. This formula is true, because of the mutual orthogonality of the impulse responses of the filters h^n and g^n and the statement of Theorem 8. From (53), (54) and Theorem 5, one can see, that

$$Z_2(\tilde{h^n})(k_n, m) = Z_2(\tilde{h^0})(2^n k_n, m) \tag{66}$$

and

$$Z_2(\tilde{g^n})(k_n, m) = Z_2(\tilde{g^0})(2^n k_n, m). \tag{67}$$

Then (65) can be rewritten as

$$Z_2(c_h^{n-1})(k_n, m) = Z_2(\tilde{h^0})(2^n k_n, -m) d_h^n[k_n] + Z_2(\tilde{g^0})(2^n k_n, -m) d_g^n[k_n], \tag{68}$$

for $k_n = 0, 1, \ldots, L/2^n - 1$ and $m = 0, 1$. This form of the formula is more convenient for the use in the algorithm.

4. If $n = 1$

   Take the inverse Zak transform of $Z_2(c_h^{n-1})$ to obtain the signal s.

   Stop.

If $n \neq 1$

   Go to step 5.

5. Set $n = n - 1$
6. Using the procedure developed in Theorem 11 (with $p = 2$) find $d_h^n[k_n] = Z_1(c_h^n)(k_n, 0)$ from $Z_2(c_h^n)$ as

$$d_h^n[k_n] = Z_2(c_h^n)(k_n \bmod (L/2^J), 0) + Z_2(c_h^n)(k_n \bmod (L/2^J), 1) e^{-j \frac{2\pi}{L/2^n} k_n}. \tag{69}$$

7. Find d_g^n as the DFT of the coefficients c_g^n.
8. Go to step 3.

There is a trivial generalization of this method to M filters operating on each resolution, applying the same procedure for $M > 2$.

7 Summary and Discussion

In this paper we presented a new approach to filtering and reconstruction. Having an arbitrary periodic synthesis filter bank, one can construct the corresponding analysis filter bank using this method even in case of undersampling. The undersampling case might be important for coding, since, as our simulations have shown, sometimes in case of the wavelet decomposition, it is better to discard low-rate subbands, instead of coding them, and allocate more bits to the high-rate ones. In cases when the wavelets are orthogonal, the reconstruction can be done by the same filter as the one used for the decomposition. Our method allows to find the biorthogonal filter pair for the case of non-orthogonal wavelets as well.

For the case of long filters a faster method for decomposition and reconstruction in the Zak transform domain was presented. Both methods can be used for performing the periodic wavelet transform and the recursive filter bank implementation.

As shown in section 4, the results presented in this paper will allow more flexibility in choosing filters for multiresolution decomposition, which is important, for example, in coding. This flexibility enabled us to find the filters that produced the best compression results. As shown in [21] this filters can be implemented much faster as recursive filters.

The generalization of the results to two-dimensional signals and images is trivial.

The paper deals mainly with periodic signal, however as shown in section 4, the whole theory can be applied to non-periodic signals as well. All of the results, however can be applied to the finite signals and images, with the assumption that they can be continued periodically. The filters can be padded with zeros to have the same length as the signals or images and also assumed to be periodic. The fact that it is assumed that the signal is periodic will definitely cause certain abnormalities at the ends of the signals or images, but the periodicity assumption is no worse and often better than the padding zeros around the signal.

There is still a certain tradeoff between the speed of the algorithm and the time-frequency localization of its impulse response. As it was mentioned before, short filters perform faster in the time domain. Use of short filters is logical for the multiresolution decomposition , since on each stage the support of the filter grows (on lower stages only close samples are being processed together, and on the higher stages more and more samples are getting involved). The solution would be to take the short filters with good localization properties and use the time domain decomposition. However, it is not necessarily true that their biorthogonal filters would also be short. In [21] it is shown that the filters biorthogonal to short filters can be implemented very fast as a combination of some short recursive filters and the original filter.

For now it seems that the most reasonable approach would be to take a short filter with good localization properties and use it for the reconstruction, while doing the decomposition in the Zak domain since for many applications e.g. digital TV, reconstruction needs to be done faster than the decomposition (unless it is a live program).

Another question that was not considered in the paper is the one of the regularity of the filters on the lower levels of resolution. One can not guarantee that applying the filter with, e.g. Gaussian response, to the lower band several times sequentially would be equivalent to passing the signal through a filter whose impulse response is regular and well behaved. That suggests using different filters on different levels of the hierarchy. This can be another direction for future research.

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APPENDIX

Proof of Theorem 1

Substituting into (4) the definitions of $Z_M(h)$ and $Z_M(s)$ according to (1) we obtain:

$$\begin{aligned}
& \frac{1}{N}\sum_{n=0}^{N-1}(\sum_{m=0}^{M-1} Z_M(h)(n,m)Z_M(s)(n,-m))e^{j\frac{2\pi}{N}ln} \\
= \; & \frac{1}{N}\sum_{n=0}^{N-1}(\sum_{m=0}^{M-1}(\sum_{i=0}^{N-1} h[m+iM]e^{-j\frac{2\pi}{N}in})(\sum_{k=0}^{N-1} s[-m+kM]e^{-j\frac{2\pi}{N}kn})e^{j\frac{2\pi}{N}ln}) \\
= \; & \frac{1}{N}\sum_{m=0}^{M-1}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1} h[m+iM]s[-m+kM](\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(l-k-i)n}) \\
= \; & \frac{1}{N}\sum_{m=0}^{M-1}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1} h[m+iM]s[-m+kM]N\delta[l-k-i], \qquad (70)
\end{aligned}$$

where $\delta[k]$ is the discrete Dirac function, which equals one at zero and zero otherwise. One can see that in the above expression the sum is non-zero only when $l-i=k$. One can rewrite the above expression as

$$\sum_{m=0}^{M-1}\sum_{i=0}^{N-1} h[m+iM]s[-m+(l-i)M] = \sum_{p=0}^{L-1} h[p]s[lM-p] = c[l]. \qquad (71)$$

Proof of Theorem 2

To prove the fact that some function $s_{\bar{h}^-}$ is the projection of the signal $s[k]$ upon the span of the functions $\{\bar{h}_l^-\}_{l=0}^{N-1}$, it is enough to show that the inner product $<(s-s_{\bar{h}^-}), \bar{h}_l^- >= 0$ for $l=0,1,\ldots,N-1$ and that $s_{\bar{h}^-}[k] \in span\{\bar{h}_l^-\}_{l=0}^{N-1}$. The inner product $<x,y>$ of any two L-periodic functions $x[k]$ and $y[k]$ is defined as

$$<x,y>=\sum_{k=0}^{L-1} x[k]\bar{y}[k]. \qquad (72)$$

In other words, one has to prove the facts:

1. that the function $s_{\bar{h}^-}$ constructed in the statement of the theorem belongs to the span of the set $\{\bar{h}_l^-\}_{l=0}^{N-1}$; and

2. that $< s, \bar{h}_l^- >=< s_{\bar{h}^-}, \bar{h}_l^- >$.

The first fact is the direct consequence of Theorem 6.

Here we prove only the second fact.

Since the Zak transform is an isometry, up to a constant factor, the fact that we need to prove is equivalent to

$$< Z_M(s), Z_M(\bar{h}_l^-) >=< Z_M(s_{\bar{h}^-}), Z_M(\bar{h}_l^-) >, \tag{73}$$

for $l = 0, 1, \ldots, N-1$.

Notice that

$$\begin{aligned} Z_M(\bar{h}_l^-) &= \sum_{i=0}^{N-1} \bar{h}[Ml - m - iM] e^{-j\frac{2\pi}{N}in} \\ &= (\sum_{i=0}^{N-1} \bar{h}[M(l-i) - m] e^{j\frac{2\pi}{N}(l-i)n}) e^{-j\frac{2\pi}{N}ln} = \bar{Z}_M(h)(n, -m) e^{-j\frac{2\pi}{N}ln}. \end{aligned} \tag{74}$$

Let us consider the right-hand side of the expression (73) and prove that it is equal to the left-hand side. Substituting the above expression (74) for $Z_M(\bar{h}_l^-)$ and the formula (7) for $Z_M(s_{\bar{h}^-})$ into (73) yields

$$\begin{aligned} &< Z_M(s_{\bar{h}^-}), Z_M(\bar{h}_l^-) >= \sum_{n=0, H_M[n]\neq 0}^{N-1} \sum_{m=0}^{M-1} \bar{Z}_M(h)(n, -m) \frac{d[n]}{H_M[n]} Z_M(h)(n, -m) e^{j\frac{2\pi}{N}ln} \\ &= \sum_{n=0, H_M[n]\neq 0}^{N-1} e^{j\frac{2\pi}{N}ln} \frac{d[n]}{H_M[n]} \sum_{m=0}^{M-1} Z_M(h)(n, -m) \bar{Z}_M(h)(n, -m) \sum_{n=0, H_M[n]\neq 0}^{N-1} e^{j\frac{2\pi}{N}ln} \frac{d[n]}{H_M[n]} H_M[n] \\ &= \sum_{n=0, H_M[n]\neq 0}^{N-1} e^{j\frac{2\pi}{N}ln} d[n]. \end{aligned} \tag{75}$$

Now, using (4) one can see that since the coefficients $d[n]$ are obtained by taking the inverse DFT of the coefficients $c[n]$, the following holds:

$$d[n] = \sum_{m=0}^{M-1} Z_M(h)(n, m) Z_M(s)(n, -m). \tag{76}$$

Substituting this expression into (75) one obtains:

$$\sum_{n=0, H_M[n]\neq 0}^{N-1} e^{j\frac{2\pi}{N}ln} \sum_{m=0}^{M-1} Z_M(h)(n, m) Z_M(s)(n, -m). \tag{77}$$

Since $H_M[n] = 0$ implies $Z_M(h)(n, m) = 0$ for all $m = 1, 2, \ldots, M-1$, these blocks may now be included in the above sum, as only terms of zero value are added. Then the above expression can be rewritten as

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}ln} \sum_{m=0}^{M-1} Z_M(h)(n, m) Z_M(s)(n, -m). \tag{78}$$

Now using (74) one obtains from the expression above:

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}ln} \sum_{m=0}^{M-1} Z_M(h)(n,m) Z_M(s)(n,-m) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} (e^{j\frac{2\pi}{N}ln} Z_M(h)(n,-m)) Z_M(s)(n,m)$$
$$= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \bar{Z}_M(\bar{h}^-)(n,m)) Z_M(s)(n,m) =< Z_M(s), Z_M(\bar{h}_l^-) > . \tag{79}$$

Proof of Theorem 3

The functions $\{h_l\}_{l=0}^{N-1}$ are linearly dependent iff there is a set $\alpha_0, \alpha_1, \ldots, \alpha_n$ of scalars, some of which are not equal to zero, such that

$$\sum_{i=0}^{N-1} \alpha_i h_i = 0. \tag{80}$$

This is equivalent to

$$\sum_{i=0}^{N-1} \alpha_i Z_M(h_i) = 0, \tag{81}$$

for any n and m. Notice that

$$Z_M(h_i)(n,m) = \sum_{k=0}^{N-1} h[m+(k-i)M] e^{-j\frac{2\pi}{N}(k-i)n} e^{-j\frac{2\pi}{N}in} = Z_M(h)(n,m) e^{-j\frac{2\pi}{N}in}. \tag{82}$$

Now we can rewrite (81) as

$$Z_M(h)(n,m) \sum_{i=0}^{N-1} \alpha_i e^{-j\frac{2\pi}{N}in} = 0. \tag{83}$$

The sum in the expression is just the DFT of the set $\{\alpha_i\}_{i=0}^{N-1}$ and cannot be zero for all n if some of the scalars are not zero. Thus $Z_M(h)(n,m)$ must be zero for every m implying that $H_M[k] = 0$. Conversely, if $H_M[n] = 0$ for some n (that means that $Z_M(h)(n,m) = 0$ for all m), we can always find such a non-zero set $\{\alpha_i\}_{i=0}^{N-1}$ that the expression (83) is zero, by choosing it to be the inverse DFT of a scalar set of size N, whose nth element is non-zero and all the other elements are zero.

Proof of Theorem 4

Using the fact that the Zak transform is linear from (9) one can obtain

$$Z_M(s_{\bar{h}^-})(n,k) = \sum_{l=0}^{N-1} \tilde{c}[l] Z_M(\bar{h}_l^-)(n,k). \tag{84}$$

Substituting the expression (74) for $Z_M(\bar{h}_l^-)$ into (84) one obtains

$$Z_M(s_{\bar{h}^-})(n,k) = \sum_{l=0}^{N-1} \tilde{c}[l]\bar{Z}_M(h)(n,-k)e^{-j\frac{2\pi}{N}ln} = \bar{Z}_M(h)(n,-k)\sum_{l=0}^{N-1} \tilde{c}[l]e^{-j\frac{2\pi}{N}ln}. \tag{85}$$

Substituting $d[n]$ in (76) into the formula for $Z_M(s_{\bar{h}^-})(n,k)$ in (7), one obtains

$$Z_M(s_{\bar{h}^-})(n,k) = \begin{cases} \bar{Z}_M(h)(n,-k)\frac{\sum_{m=0}^{M-1} Z_M(h)(n,m)Z_M(s)(n,-m)}{H_M[n]} & \text{if } H_M[n] \neq 0 \\ 0 & \text{if } H_M[n] = 0, \end{cases}. \tag{86}$$

Comparing the above two formulas one can see that the coefficients $\tilde{c}[l]$ can be chosen to be the inverse DFT of the sequence $\sum_{m=0}^{M-1} \frac{Z_M(h)(n,m)}{H_M[n]} Z_M(s)(n,-m)$, which gives the expression (10). When the set $\{\bar{h}_l^-\}_{l=0}^{N-1}$ is linearly independent, the choice is unique.

Proof of Theorem 5

Applying equation (4) to the filter, the M-Zak transform of whose impulse response is given by (12) one obtains

$\frac{1}{N}\sum_{n=0,H_M[n]=0}^{N-1}(\sum_{m=0}^{M-1} \frac{Z_M(h)(n,m)}{H_M[n]} Z_M(s)(n,-m))e^{j\frac{2\pi}{N}ln}$.

But this is precisely the expression (10) for the coefficients $\tilde{c}[l]$.

Proof of Lemma 1

$$Z_M(w_p)(n,m) = \sum_{i=0}^{N-1} w[m+iM]e^{-j\frac{2\pi}{N}in} = \sum_{i=0}^{N-1}\sum_{l=0}^{N-1} h[m+(i+l)M]e^{-j\frac{2\pi}{N}pl}e^{-j\frac{2\pi}{N}in}$$

$$= \sum_{s=0}^{N-1} h[m+sM]e^{-j\frac{2\pi}{N}sn}\sum_{l=0}^{N-1} e^{-j\frac{2\pi}{N}l(n-p)} = Z_M(h)(n,m)N\delta[n-p]. \tag{87}$$

Proof of Theorem 6

First, applying (16) construct the functions $\{w_p^s\}_{p=0,s=0}^{N-1,S-1}$ (subscript here does not denote the M-shift) according to the formula

$$w_p^s[k] = Z_M(h^s)(p,k). \tag{88}$$

Because of the lemma, we know that

$$Z_M(w_p^s)(n,m) = Z_M(h^s)(n,m)N\delta[n-p]. \tag{89}$$

Combining (20), (18) and (89), one can write:

$$Z_M(f)(n,m) = \sum_{s=0}^{S-1} a_{s,n} Z_M(h^s)(n,m)$$
$$= \frac{1}{N}\sum_{s=0}^{S-1}\sum_{p=0}^{N-1} a_{s,n} Z_M(w_p^s)(n,m) = \frac{1}{N}\sum_{s=0}^{S-1} a_{s,n} Z_M(w_n^s)(n,m). \tag{90}$$

Notice that because of (89), one can write:

$$\sum_{p=0}^{N-1} a_{s,p} Z_M(w_p^s)(n,m) = a_{s,n} Z_M(w_n^s)(n,m). \tag{91}$$

Substituting into (91) one obtains: $Z_M(f)(n,m) = \frac{1}{N}\sum_{s=0}^{S-1}\sum_{p=0}^{N-1} a_{s,p} Z_M(w_p^s)(n,m)$. In this expression the coefficients $a_{s,p}$ do not depend on the block number. Therefore the inverse Zak transform operating on both sides of the equation yields

$$f[k] = \frac{1}{N}\sum_{s=0}^{S-1}\sum_{p=0}^{N-1} a_{s,p} w_p^s[k]. \tag{92}$$

Therefore: $f \in span(\{w_p^s\}_{p=0,s=0}^{N-1,S-1})$. Because of (19), we know that $span(\{w_p^s\}_{p=0,s=0}^{N-1,S-1}) \in span(\{h_n^s\}_{n=0,s=0}^{N-1,S-1})$ and therefore $f \in span(\{h_n^s\}_{n=0,s=0}^{N-1,S-1})$.

Proof of the converse statement is easier: indeed, suppose that

$$f[k] = \sum_{s=0}^{S-1}\sum_{l=0}^{N-1} b_{s,l} h_l^s[k] = \sum_{s=0}^{S-1}\sum_{l=0}^{N-1} b_{s,l} h^s[k-lM]. \tag{93}$$

By taking the M-Zak transform of the both parts and using $Z(h_l)(n,m) = Z(h)(n,m)e^{-j\frac{2\pi}{N}ln}$, one obtains:

$$Z_M(f)(l,m) = \sum_{s=0}^{S-1}\sum_{l=0}^{N-1} b_{s,l} Z(h^s)(n,m)e^{-j\frac{2\pi}{N}ln} = \sum_{s=0}^{S-1}(\sum_{l=0}^{N-1} b_{s,l}e^{-j\frac{2\pi}{N}ln})Z(h^s)(n,m). \tag{94}$$

Now taking $a_{s,l} = \sum_{l=0}^{N-1} b_{s,l}e^{-j\frac{2\pi}{N}ln}$ we arrive at equation (20).

Proof of Theorem 7

Note that because of the invariance of the sets with respect to simultaneous M-shifts of all their functions, the statement that the sets are orthogonal in the sense of the theorem above is equivalent to the statement that h^0 is orthogonal to any function of the second set. Since the Zak Transform is a unitary (up to a scalar) transformation, the orthogonality of the functions is equivalent to the orthogonality of their Zak transforms. Then

$$\sum_{k=0}^{L-1} h^0[k]\bar{h}_l^1[k] = \sum_{n,m} Z_M(h^0)(n,m)\bar{Z}_M(h_l^1)(n,m)$$
$$= \sum_{n,m} Z_M(h^0)(n,m)\bar{Z}_M(h^1)(n,m)e^{j\frac{2\pi}{N}ln} = \sum_n e^{j\frac{2\pi}{N}ln}\sum_m Z_M(h^0)(n,m)\bar{Z}_M(h^1)(n,m) \tag{95}$$

The above sum is the inverse DFT of the terms $\sum_m Z_M(h^0)(n,m)\bar{Z}_M(h^1)(n,m)$, taken as a function of n and therefore the sum equals zero for every l iff every such term is zero.

Proof of Theorem 8

The rows of $\boldsymbol{V}$ are linear combinations of the rows of $\boldsymbol{H}$. Theorem 6 shows that every function v^q belongs to the span of the set $\{h_l^{q-}\}_{l=0,q=0}^{N-1,K-1}$. Since $\boldsymbol{O}$ is invertible, the inverse is also true: every function h^q belongs to the span of the set $\{v_l^q\}_{l=0,q=0}^{N-1,K-1}$. Therefore the sets have the same span, and so do the conjugate sets $\{\bar{v}_l^{q-}\}_{l=0,q=0}^{N-1,K-1}$ and $\{\bar{h}_l^{q-}\}_{l=0,q=0}^{N-1,K-1}$.

Because of Theorem 7, sets $\{\bar{v}_l^{0-}\}_{l=0}^{N-1}, \{\bar{v}_l^{1-}\}_{l=0}^{N-1}, \ldots, \{\bar{v}_l^{K-1-}\}_{l=0}^{N-1}$ are mutually orthogonal. Therefore to find the projection upon the span of $\{\bar{v}_l^{q-}\}_{l=0,q=0}^{N-1,K-1}$ one can find the projections of the signal s upon each of the sets separately and sum up the results, which is exactly the expression in (24). Since the sets $\{\bar{v}_l^{q-}\}_{l=0,q=0}^{N-1,K-1}$ and $\{\bar{h}_l^{q-}\}_{l=0,q=0}^{N-1,K-1}$ have the same span, the projection upon the span of $\{\bar{h}_l^{q-}\}_{l=0,q=0}^{N-1,K-1}$ is also found.

Proof of Theorem 9

Suppose that the coefficients $\tilde{c}^q[l]$ were found, then projection $s_{\bar{h}^-}$ can be expressed as:

$$s_{\bar{h}^-}[k] = \sum_{q=0}^{K-1}\sum_{l=0}^{N-1} \tilde{c}^q[l]\bar{h}_l^{q-}[k]. \tag{96}$$

Since the Zak Transform is unitary up to a scalar, the above expression is equivalent to

$$Z_M(s_{\bar{h}^-})(n,m) = \sum_{q=0}^{K-1}\sum_{l=0}^{N-1} \tilde{c}^q[l] Z_M(\bar{h}_l^{q-})(n,m) = \sum_{q=0}^{K-1} \bar{Z}_M(h^q)(n,-m)\tilde{d}^q[n]. \tag{97}$$

Equating (24) and (97),

$$\sum_{q=0}^{K-1} \bar{Z}_M(h^q)(n,-m)\tilde{d}^q[n] = \sum_{q=0}^{K-1} \bar{Z}_M(v^q)(n,-m)\tilde{w}^q[n]. \tag{98}$$

Let us introduce some new notations:

$$\boldsymbol{H}^-[n] = \begin{pmatrix} Z_M(h^0)(n,0) & Z_M(h^0)(n,-1) & ,\ldots, & Z_M(h^0)(n,-(M-1)) \\ Z_M(h^1)(n,0) & Z_M(h^1)(n,-1) & ,\ldots, & Z_M(h^1)(n,-(M-1)) \\ \cdots & \cdots & \cdots & \cdots \\ Z_M(h^{K-1})(n,0) & Z_M(h^{K-1})(n,-1) & ,\ldots, & Z_M(h^{K-1})(n,-(M-1)) \end{pmatrix},$$

$$\boldsymbol{V}^-[n] = \begin{pmatrix} Z_M(v^0)(n,0) & Z_M(v^0)(n,-1) & ,\ldots, & Z_M(v^0)(n,1(M-1)) \\ Z_M(v^1)(n,0) & Z_M(v^1)(n,q1) & ,\ldots, & Z_M(v^1)(n,-(M-1)) \\ \cdots & \cdots & \cdots & \cdots \\ Z_M(v^{K-1})(n,0) & Z_M(v^{K-1})(n,-1) & ,\ldots, & Z_M(v^{K-1})(n,-(M-1)) \end{pmatrix}.$$

One can then express (98) in vector notation as:

$$\tilde{\boldsymbol{D}}[n]\bar{\boldsymbol{H}}^{-}[n] = \tilde{\boldsymbol{W}}[n]\bar{\boldsymbol{V}}^{-}[n]. \tag{99}$$

As a consequence of (22),

$$\bar{\boldsymbol{V}}^{-}[n] = \bar{\boldsymbol{O}}[n]\bar{\boldsymbol{H}}^{-}[n]. \tag{100}$$

Then substituting (100) into (99) one obtains the following:

$$\tilde{\boldsymbol{D}}[n]\bar{\boldsymbol{H}}^{-}[n] = \tilde{\boldsymbol{W}}[n]\bar{\boldsymbol{O}}[n]\bar{\boldsymbol{H}}^{-}[n]. \tag{101}$$

This means that to obtain the projection one can choose the vector $\tilde{\boldsymbol{D}}[n]$ as in (27), which proves the first statement.

One can see, also, that (25) can be rewritten in the matrix form as

$$\tilde{\boldsymbol{W}}^{T}[n] = (\boldsymbol{V}[n]\boldsymbol{V}^{*}[n])^{-1}\boldsymbol{V}[n]\boldsymbol{S}^{-}[n]. \tag{102}$$

Indeed since $\boldsymbol{V}^{-}$ has orthogonal rows, $\boldsymbol{V}[n]\boldsymbol{V}^{*}[n]$ is a $K \times K$ diagonal matrix $\boldsymbol{V}[n]\boldsymbol{V}^{*}[n] = diag[V^{0}[n], V^{1}[n], \ldots, V^{K-1}[n]]$, so that the term $(\boldsymbol{V}[n]\boldsymbol{V}^{*}[n])^{-1}$ takes care of the division in (25). Substituting (22) into (102) one derives that

$$\begin{aligned}\tilde{\boldsymbol{W}}^{T}[n] &= (\boldsymbol{O}[n]\boldsymbol{H}[n]\boldsymbol{H}^{*}[n]\boldsymbol{O}^{*}[n])^{-1}\boldsymbol{O}[n]\boldsymbol{H}[n]\boldsymbol{S}^{-}[n] \\ &= (\boldsymbol{O}^{*}[m])^{-1}(\boldsymbol{H}[n]\boldsymbol{H}^{*}[n])^{-1}\boldsymbol{H}[n]\boldsymbol{S}^{-}[n]. \end{aligned} \tag{103}$$

Substitution of the above expression into (27) and straightforward matrix manipulations produce (28). The second statement is proved.

Proof of Theorem 10

Combining (28), (30) and (29) one obtains

$$\tilde{c}^{q}[l] = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} Z(\tilde{h}^{q})(n,m)Z(s)(n,-m)e^{j\frac{2\pi}{N}nl}, \tag{104}$$

which is exactly what one would obtain by passing the signal through a filter with impulse response $\tilde{h}^{q}$ as seen in equation (4).

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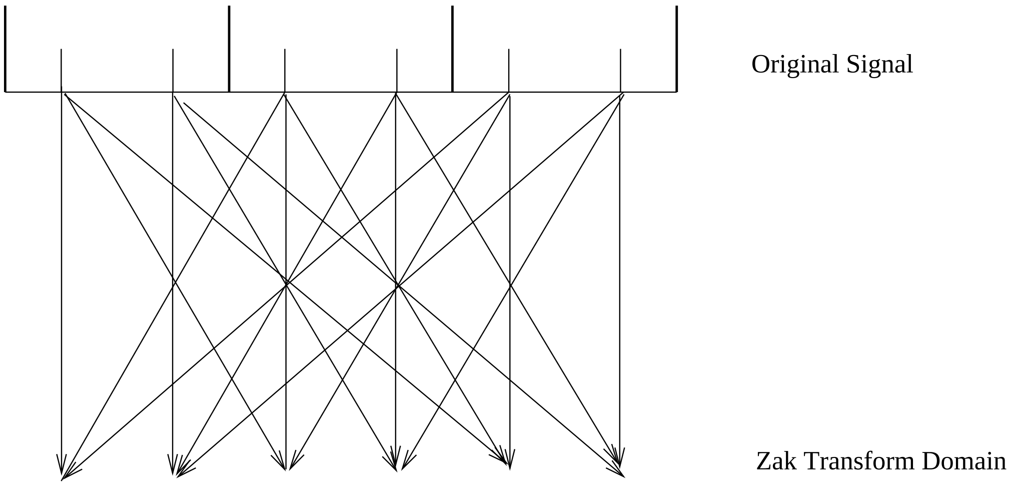

Figure 1: Zak transform of a signal of period $L = 6$ with the number of blocks $N = 3$ and the number of samples in each block $M = 2$. The thicker line segments denote the separations between the blocks and the thinner ones, the signal samples.

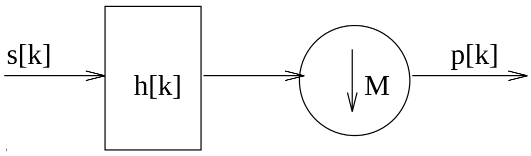

Figure 2: Decimator

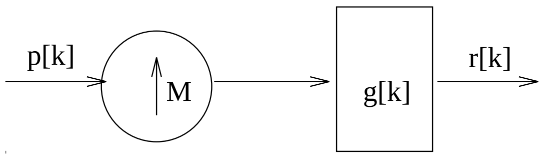

Figure 3: Interpolator

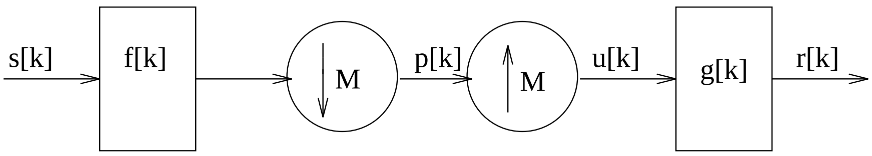

Figure 4: Single Filter Structure

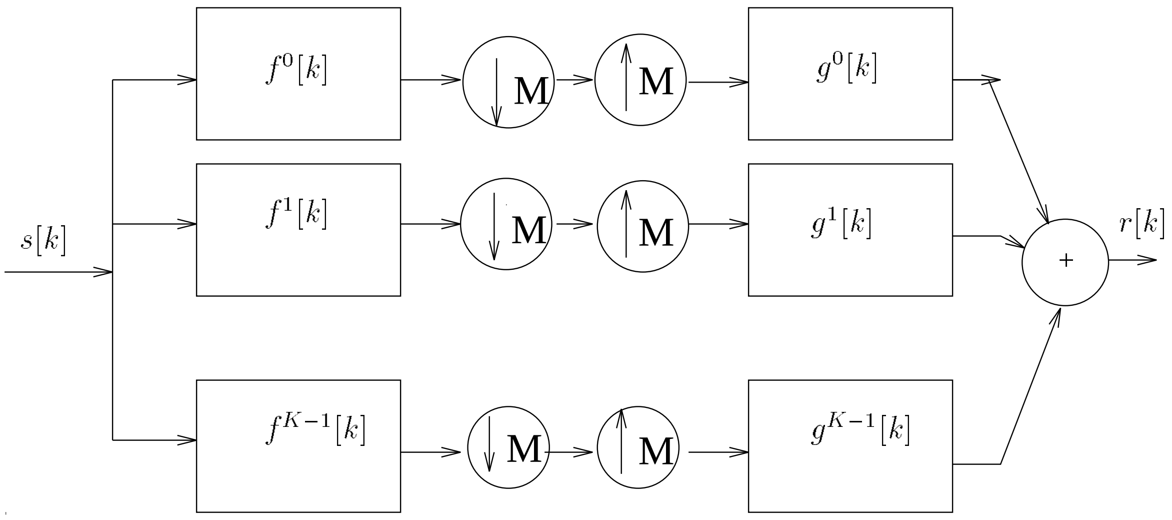

Figure 5: Filter Bank

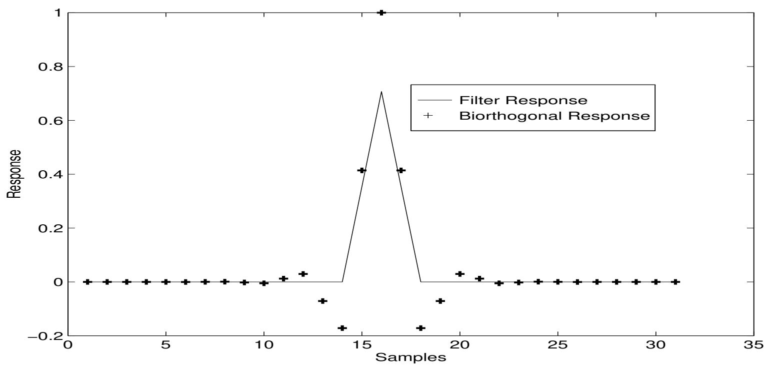

Figure 6: A Plot of the normalized impulse response of the filter with tap values [1 2 1] and its Biorthogonal Filter

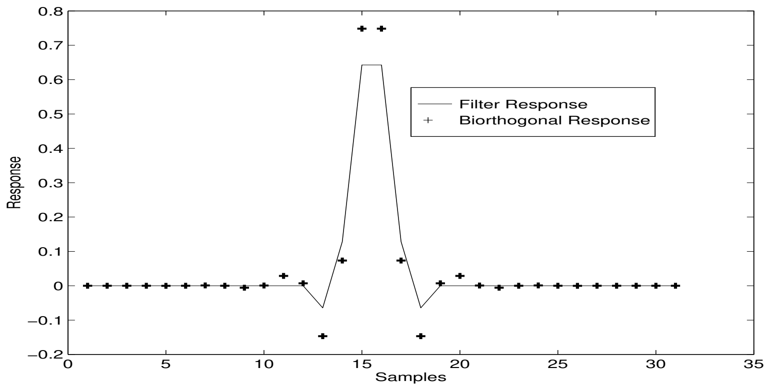

Figure 7: A Plot of the normalized impulse response of the filter with tap values [-1 2 10 10 2 -1] and its Biorthogonal Filter

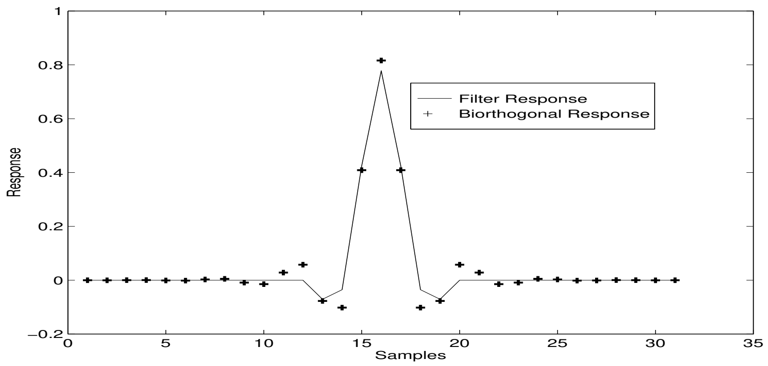

Figure 8: A Plot of the normalized impulse response of the filter with tap values [-1 -0.5 6 11 6 -0.5 -1] and its Biorthogonal Filter

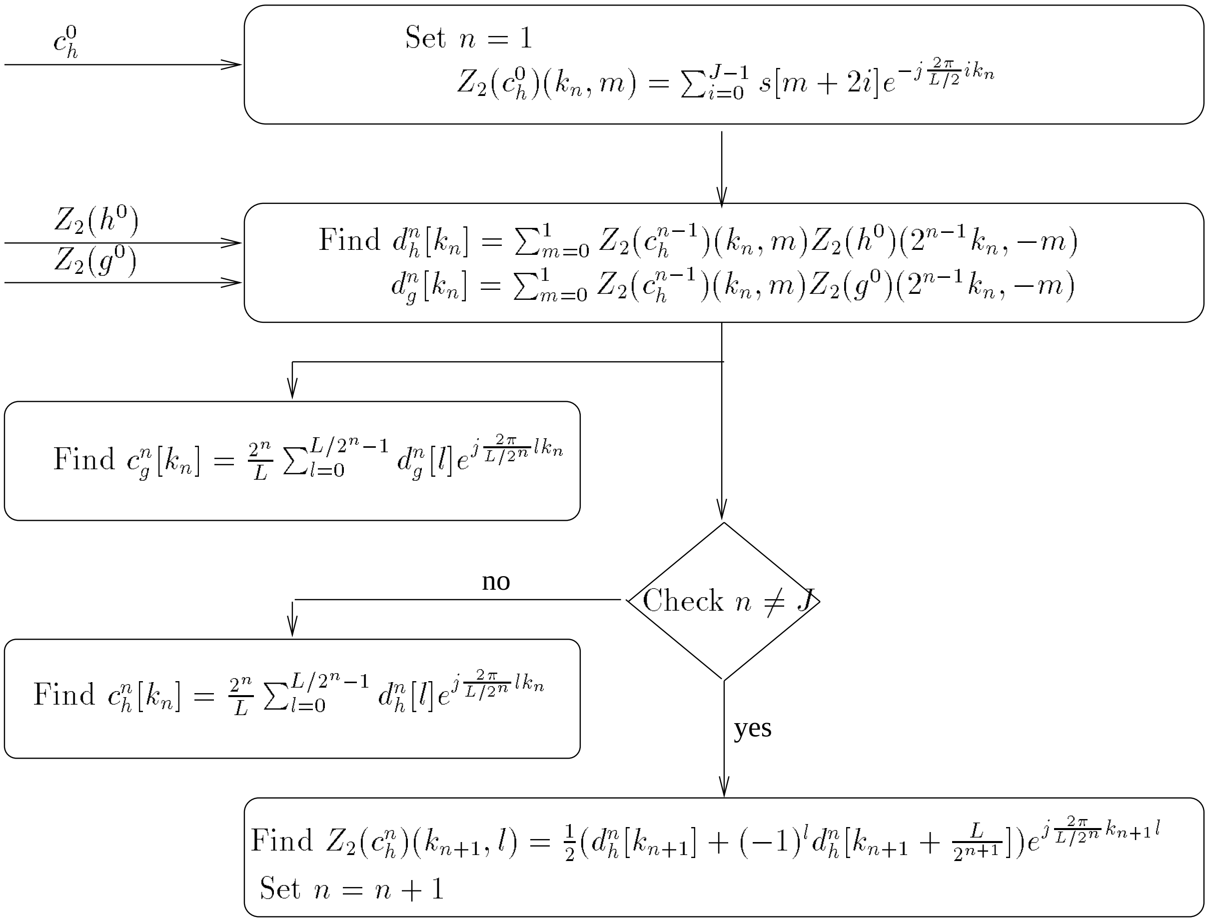

Figure 9: The Fast Wavelet Transform Diagram

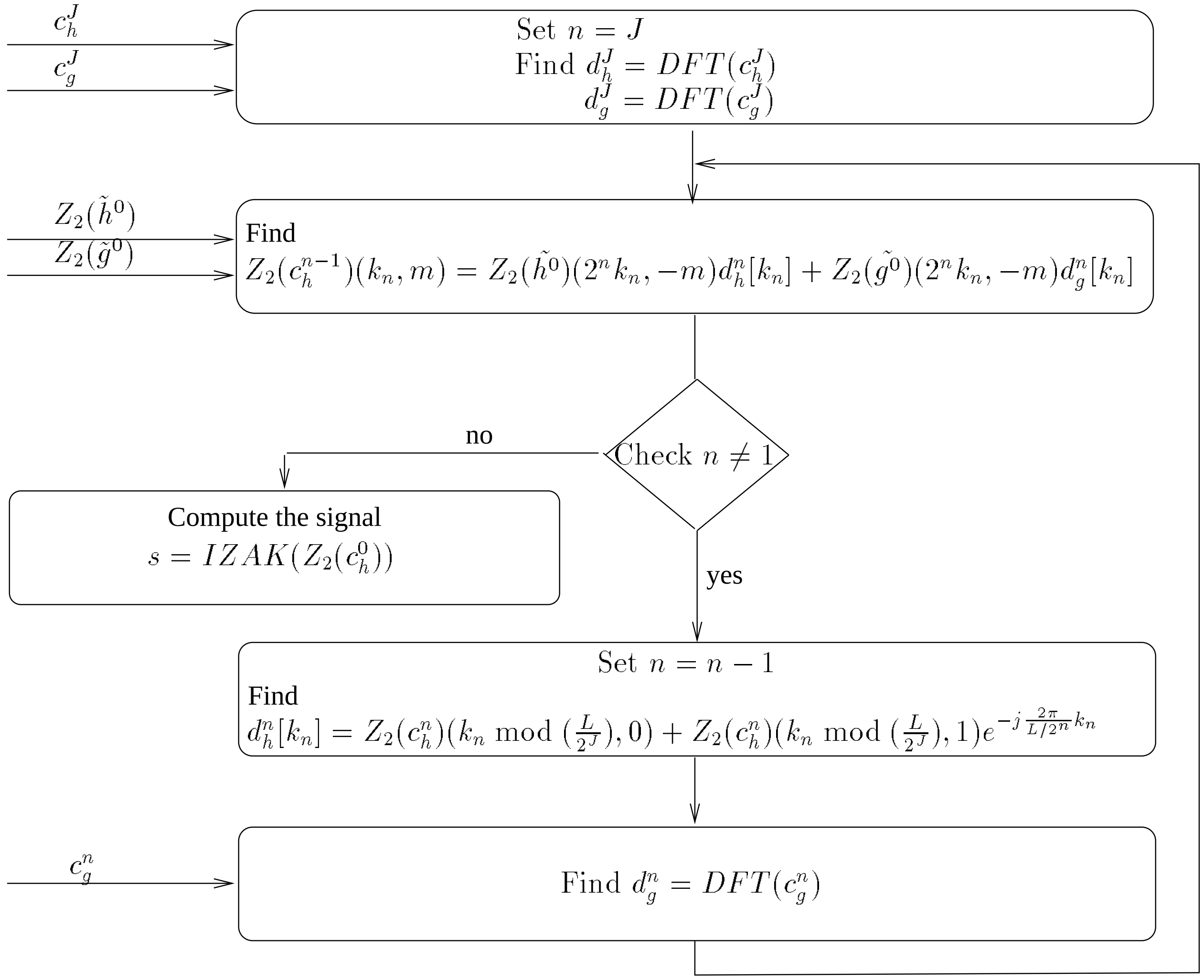

Figure 10: The Fast Wavelet Transform Diagram. DFT - Discrete Fourier Transform, IZAK - inverse Zak Transform