EEM3L1: Numerical and Analytical Techniques

Lecture 5:

Singular Value Decomposition

SVD (1)

Motivation for SVD (1)

SVD = Singular Value Decomposition

Consider the system of linear equations $Ax = b$

Suppose b is perturbed to $b+\delta b$

Solution becomes $x = A^{-1}b + A^{-1}\delta b$

The consequent change in x is therefore $A^{-1}\delta b$

For what perturbation δb will the error be biggest?

How big can the norm of the error be, in terms of $\|\delta b\|$?

The norm of the error, relative to $\|\delta b\|$ can be expressed in terms of a number called the **smallest singular value** of A

Motivation for SVD (2)

Which direction b must be perturbed in to give the biggest error?

If *cond(A)* is large. How can we then find an accurate solution to $Ax = b$?

Both of these questions can also be addressed using Singular Value Decomposition

The remainder of the section of linear algebra will be taken up with **Singular Value Decomposition (SVD)**

Orthogonal Matrices revisited

Remember that an m x n matrix U is called **column orthogonal** if $U^T U = I$, where I is the identity matrix

In other words, the column vectors in U are orthogonal to each other and each of them are of unit norm

If $n = m$ then U is called **orthogonal**.

In this case $UU^T = I$ also

SVD of a Matrix

Let A be an m x n matrix such that the number of rows m is greater than or equal to the number of columns n. Then there exists:

(i) an m x n **column orthogonal** matrix U

(ii) an n x n **diagonal** matrix S, with positive or zero elements, and

(iii) an n x n orthogonal matrix V

such that: $A = USV^T$

This is the **Singular Value Decomposition (SVD)** of A

The Singular Values of *A*

Suppose $S = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma\text{-}_n\}$.

By convention it is assumed that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$

The values $\sigma_1, \sigma_2, \ldots, \sigma\text{-}_n$ are called the **singular values** of *A*

SVD of a square matrix

The case where A is an n x n square matrix is of partiicular interest.

In this case, the **Singular Value Decomposition** of A is given

$A = USV^T$

Where V and U are **orthogonal** matrices

Example

$$\text{Let } A = \begin{bmatrix} 0.9501 & 0.8913 & 0.8214 & 0.9218 \\ 0.2311 & 0.7621 & 0.4447 & 0.7382 \\ 0.6068 & 0.4565 & 0.6154 & 0.1763 \\ 0.4860 & 0.0185 & 0.7919 & 0.4057 \end{bmatrix}$$

In MATLAB, >> [U,S,V]=svd(A); returns the SVD of *A*

$$U = \begin{bmatrix} 0.7301 & 0.1242 & 0.1899 & -0.6445 \\ 0.4413 & 0.6334 & -0.3788 & 0.5104 \\ 0.3809 & -0.3254 & 0.6577 & 0.5626 \\ 0.3564 & -0.6910 & -0.6229 & 0.0871 \end{bmatrix} \quad V = \begin{bmatrix} 0.4903 & -0.4004 & 0.5191 & -0.5743 \\ 0.4770 & 0.6433 & 0.4642 & 0.3783 \\ 0.5362 & -0.5417 & -0.2770 & 0.5850 \\ 0.4945 & 0.3638 & -0.6620 & -0.4299 \end{bmatrix}$$

$$S = \begin{bmatrix} 2.4479 & 0 & 0 & 0 \\ 0 & 0.6716 & 0 & 0 \\ 0 & 0 & 0.3646 & 0 \\ 0 & 0 & 0 & 0.1927 \end{bmatrix}$$

Singular values and Eigenvalues

The singular values of A are **not** the same as its eigenvalues

>> eig(A)

ans =

2.3230

0.0914+0.4586i

0.0914-0.4586i

0.2275

For any matrix A the matrix $A^H A$ is normal with non-negative eigenvalues.

The **singular values of A are the square roots of the eigenvalues of $A^H A$**

Calculating Inverses with SVD

Let *A* be an *n* x *n* matrix.

Then *U*, *S* and *V* are also *n* x *n*.

U and *V* are orthogonal, and so their inverses are equal to their transposes.

S is diagonal, and so its inverse is the diagonal matrix whose elements are the inverses of the elements of *S*.

$$A^{-1} = V\left[diag(\sigma_1^{-1}, \sigma_2^{-1}, ..., \sigma_n^{-1})\right]U^T$$

Calculating Inverses (contd)

If one of the σ_is is zero, or so small that its value is dominated by round-off error, then there is a problem!

The more of the σ_is that have this problem, the 'more singular' *A* is.

SVD gives a way of determining how singular *A* is.

The concept of 'how singular' *A* is is linked with the condition number of *A*

The **condition number of *A* is the ration of its largest singular value to its smallest singular value**

The Null Space of *A*

Let *A* be an *n* x *n* matrix

Consider the linear equations $Ax=b$, where *x* and *b* are vectors.

The set of vectors *x* such that $Ax=0$ is a linear vector space, called the **null space** of *A*

If *A* is invertible, the null space of *A* is the zero vector

If *A* is singular, the null space will contain non-zero vectors

The dimension of the null space of *A* is called the **nullity** of *A*

The Range of *A*

The set of vectors which are 'targets' for A, i.e. the set of all vectors b for which there exists a vector x such that $Ax=b$ is called the **range** of A

The range of A is a linear vector space whose dimension is the **rank** of A

If A is singular, then the rank of A will be less than n

$n = Rank(A)+Nullity(A)$

SVD, Range and Null Space

Singular Valued Decomposition constructs orthonormal bases for the range and null space of a matrix

The columns of *U* which correspond to non-zero singular values of *A* are an **orthonormal set of basis vectors for the range of *A***

The columns of *V* which correspond to zero singular values form an **orthonormal basis for the null space of *A***

Solving linear equations with SVD

Consider a set of **homogeneous** equations $\boldsymbol{Ax=0}$.

Any vector x in the null space of A is a solution.

Hence any column of V whose corresponding singular value is zero is a solution

Now consider $\boldsymbol{Ax=b}$ **and** $\boldsymbol{b \neq 0}$,

A solution only exists if b lies in the range of A

If so, then the set of equations does have a solution.

In fact, it has infinitely many solutions because if x is a solution and y is in the null space of A, then $x+y$ is also a

Solving $Ax=b\neq 0$ using SVD

If we want a particular solution, then we might want to pick the solution x with the smallest length $|x|^2$

Solution is

$$x = V\left[diag(\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_n^{-1})\right](U^T b)$$

where, for each singular value σ_j such that $\sigma_j=0$, $\boldsymbol{\sigma_j^{-1}}$ **is replaced by *0***

Least Squares Estimation

If b is not in the range of A then there is no vector x such that $Ax=b$. So

$$x = V\left[diag(\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_n^{-1})\right](U^T b)$$

cannot be used to obtain an exact solution.

However, the vector returned will do the 'closest possible job' in the least squares sense.

It will find the vector x which minimises $R = \|Ax - b\|$

R is called the **residual** of the solution