Mean Field Annealing Deformable Contour Method: A Constrained Global Optimization Approach

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Abstract

This paper presents an efficient global optimization approach to the problem of constrained contour energy minimization for object boundary extractions. In the method, with a given contour energy function, different target boundaries can be modeled as constrained global optimal solutions under different constraints expressed as a set of parameters characterizing the target contour interior structures. To search for the constrained global optimal solutions, a fast and efficient global approach based on mean field annealing (MFA) is employed to avoid local minima. As an illustrative example, three target boundaries in a synthetic image are modeled as constrained global energy minimum contours with different constraint parameters and are successfully located using the derived algorithm. A conventional variational based deformable contour method [1] with the same energy function and constraint fails to achieve the same task. Experimental evaluations and comparisons with other methods on ultrasound pig heart, MRI knee, and CT kidney images where gaps, blur contour segments having complex shape and inhomogeneous interiors have been conducted with most favorable results.

1. Introductions

Deformable contour methods (DCMs), since originated by Kass et al [2], receive tremendous amount of attentions. Typically, most DCMs [2] [4] model boundary extraction as a contour energy minimization problem. The difficulty is that the contour energy functions are often functions of image gradient and have multiple energy minima. As a result, undesired local energy minimum contours instead of target boundary are often resulted. To handle this problem, improvements in DCMs can be divided into two categories. In the first category, the methods [1] [5] propose new energy functions integrating both region features and image gradient information to alleviate the problem of multiple energy minima. However, in challenging situations, where a combination of region and image gradient information fails to accurately identify the target boundary, the methods still tends to be trapped into undesired local energy minima. The second category of DCMs model target boundary as global energy minimum [10] [11] and take global optimization approaches specifically simulated annealing to locate them. The difficulty is that in a complex image context, the target boundary is usually a global energy minimum under certain constraints (for instance, constraints of target object interior characteristics) instead of the actual global energy minimum contour. Thus in [10] [11], either a major modification of the energy function or a preset mask [11] constraining the contour searching space within a neighborhood close to target boundary is often required.

In this paper, we model target boundary as global contour energy minimum under a constraint of region features. We take mean field annealing approach (MFA), which is a deterministic approach and requires much less computational complexity than simulated annealing, to locate the constrained global optimal solution. Favorable results on challenging segmentation problems are then reported.

In Section 2 and 3, MFA theory and the problem formulation are introduced and discussed. Section 4 details the derivation of the proposed approach. Illustrative examples are shown in Section 5 while applications are demonstrated in Section 6. In section 7, the conclusion is provided.

2. A Brief Review of Mean Field Annealing

Mean field annealing is a global optimization method derived from statistical mechanics [7]. Let $f$ be a random variable and $E(f)$ be an energy function to be minimized. Without any prior knowledge, the probability distribution of $f$ is assumed to be Gibbs distribution,

$$ P(f) = \frac{1}{Z} \exp\left(-\frac{1}{T} E(f)\right) $$  \hspace{1cm} (1) $$

with the partition function $Z$ being

$$ Z = \sum_{f \in \mathcal{F}} \exp\left(-\frac{1}{T} E(f)\right) $$  \hspace{1cm} (2)
where \( P^* \) is all the possible configurations of \( f \) and \( T \) is the temperature. The statistical mean of \( f \) at temperature \( T \) is defined as,

\[
\bar{f}(T) = \frac{1}{Z} \sum_{f \in \Omega} fP(f) = 1 \exp \left( -\frac{1}{T}E(f) \right)
\]

According to mean field annealing theory [7], \( \bar{f}(T) \) is of importance due to the well known fact that as the temperature approaches zero, \( \bar{f}(T) \) approaches the global optimal point \( f^* \),

\[
\lim_{T \to 0} \bar{f}(T) = \sum_{f \in \Omega} fP(f) = f^*
\]

This suggests that instead of minimizing \( E(f) \) directly, we can try to evaluate mean field \( \bar{f}(T) \) and then track \( f^* \). Note that in many applications, \( f \) is a random function of \( t \) rather than a random variable. In these cases, the mean of \( f \) at temperature \( T \) is also a function of \( t \).

3. Problem Formulation

Problem statement

Let \( \Phi \) be an open domain subset of \( \mathbb{R}^2 \) and \( I(x,y): \Phi \to \mathbb{R} \) be the image intensity function. Consider a target object with boundary \( \Gamma(q) \) and interior \( \Omega_q \) in the image. The average image intensity in \( \Omega_q \), \( I_o \), and the variance of image intensity in \( \Omega_q \), \( \sigma^2 \), can be determined as

\[
I_o = \frac{\int_{\Omega_q} I(x,y) \, dx \, dy}{\int_{\Omega_q} \, dx \, dy},
\]

and

\[
\sigma^2 = \frac{\int_{\Omega_q} (I(x,y) - I_o)^2 \, dx \, dy}{\int_{\Omega_q} \, dx \, dy}.
\]

We can model target boundary \( \Gamma(q) \) as a close contour having the global minimum energy,

\[
E_c(\Gamma(q)) = \iint_{\Gamma(q)} \|\nabla I(\Gamma(q))\| \, ds \approx \iint_{\Gamma(q)} \frac{1}{1 + \|\nabla I(\Gamma(q))\|^2} + \alpha \, ds
\]

satisfying

\[
D(x,y) = \exp \left[ -\frac{(I(x,y) - I_o)^2}{2\sigma^2} \right] \geq T_v \quad \text{if} \quad (x,y) \in \Omega_q \quad \text{(6)}
\]

where \( s \) is the normalized arc length, \( q \) is a contour parameter, \( g(\|\nabla I(\Gamma(q))\|) \) is a contour energy function firstly introduced in [4], which can be any positive decreasing function. \( \|\nabla I(\Gamma(q))\| \) is the gradient of \( I(x,y) \) with \( (x,y) \in \Gamma(q), \alpha > 0 \) is a constant. \( D(x,y) \) is used to characterize \( \Omega_q \). \( T_v > 0 \) is a threshold that can be adjusted for modeling different target boundaries as constrained global energy minima. A large \( T_v \) often indicates a small admissible set of contours that satisfy the constraint of Eq. (6) and vise versa. Other \( D(x,y) \) and energy function can also be considered.

Then our problem is to estimate the optimal values of \( \sigma^2 \), \( I_o \) and then find a close contour \( C(q,t) \) enclosing region \( \Omega_q(t) \) at time \( t \), such that

\[
E_c(C(q,t)) = \int_{\Omega_q} g(\|\nabla I(C(q,t))\|) \, ds
\]

is global minimum under the constraint,

\[
D(x,y) \geq T_v \quad \text{for} \quad (x,y) \in \Omega_q(c(t)) \quad \text{(8)}
\]

The Lagrange formulation of Eq. (7) is

\[
L(C(q,t),\sigma^2,I_o) = \int_{\Omega_q} g(\|\nabla I(C(q,t))\|) \, ds
\]

\[
-\lambda_c \int_{\Omega_q} D(x,y) - T_v \, dx \, dy \quad \text{(9)}
\]

where \( \lambda_c > 0 \) is a Lagrange multiplier. Since \( \Gamma(q) \) is the constrained global energy minimum, according to Lagrange multiplier condition, the optimal setting of \( \lambda_1, \lambda^*_1 \), should satisfy

\[
\frac{\partial L(C(q,t),\sigma^2,I_o,\lambda_1)}{\partial C(q,t)} = \frac{\partial L(C(q,t),\sigma^2,I_o,\lambda^*_1)}{\partial C(q,t)} = 0
\]

According to [1],

\[
\frac{\partial L(C(q,t),\sigma^2,I_o,\lambda_1)}{\partial C(q,t)} = \lambda_1 [D(x,y) - T_v] + k \|\nabla I\| - \nabla g \cdot \vec{N}
\]

where \( k \) is contour curvature and \( \vec{N} \) is the normal direction of \( C(q,t) \). Thus satisfying Eq. (10) is equivalent to satisfying

\[
F(\Gamma(q),\sigma^2,I_o,\lambda_1) = 0
\]

In implementations, it may be difficult to find a \( \lambda_1^* \) satisfying Eq. (11). As an alternative, we search for a \( \lambda_1^* \) such that
\[ \left\| F(\Gamma(q),\sigma^2,I_o,\lambda^*_1) \right\|_\mu = \min \left\{ F(\Gamma(q),\sigma^2,I_o,\lambda) \right\}_\mu \]  
(12)

where \( \lambda > 0 \), \( \| \cdot \|_\mu \) is the \( \mu \) norm function of \( a \) and \( \mu > 0 \). To simplify the computation of Eq. (12), we often choose \( \mu = \infty \), and \( \lambda^*_1 \) can then be determined. With this, Eq. (9) can be finally written as

\[ L(C(q,t),\sigma^2,I_o,\lambda^*_1) = \oint_{\Gamma} g(\nabla I(C(q,t))) ds - \lambda^*_1 \int_{\Omega_0} (D(x,y) - T_x) dxdy \]  
(13)

To minimize Eq. (13), the original constrained optimization deformable contour method (CODCM) [1] initially estimates \( \sigma^2, I_o, \lambda^*_1 \) and then minimizes \( L(C(q,t),\sigma^2,I_o,\lambda^*_1) \) using a variational approach. The contour evolution formula is

\[ \frac{\partial C(q,t)}{\partial t} = \left\{ \lambda^*_1 \left[ D(x,y) - T_x \right] + \lambda g(\nabla I) - \nabla g \cdot \nabla I \right\} N \]  
(14)

Since \( E_t(.) \) is nonconvex having multiple local energy minima and \( \lambda^*_1 \) is often small, Eq. (14) tends to be trapped into local energy minima. An extra constant balloon force \( b \) has to be added,

\[ \frac{\partial C(q,t)}{\partial t} = \left\{ \lambda^*_1 \left[ D(x,y) - T_x \right] + b g(\nabla I) - \nabla g \cdot \nabla I \right\} N \]  
(15)

In Eq. (15), even with a small \( b \), \( L(C(q,t),\sigma^2,I_o,\lambda^*_1) \) is still nonconvex and has multiple minima while with a large \( b \), locating the global minimum of \( L(C(q,t),\sigma^2,I_o,\lambda^*_1) \) is equivalent to locating a local energy minimum near the maximum of \( \int_{\Omega_0} (D(x,y) - T_x) dxdy \), which is not necessarily the constrained global energy minimum.

4. Derivation of the Approach

Energy function formulation

To overcome the difficulties of CODCM [1] [16], we add an extra constraint \( E_p - T_p \leq 0 \) that can convexify Eq. (13) without deviating the global optimal solutions, where

\[ E_p = (\sigma^2 - \sigma^2_c)^2 + (I_o - I^*_o)^2 + (\lambda^*_1 - \lambda^*_1)^2 \]  
(16)

\( T_p \geq 0 \) is a small constant. \( \sigma^2_c, I^*_o, \) and \( \lambda^*_1 \) are obtained from \( \Omega_c(t) \) as follows:

\[ I^*_o = \frac{\int_{\Omega_0} I(x,y) dxdy}{\int_{\Omega_0} dxdy} \]  
(17)

\[ \sigma^2_c = \frac{\int_{\Omega_0} I(x,y) dxdy}{\int_{\Omega_0} dxdy} \]  
(18)

Then Eq. (15) can be written as,

\[ l_p(C(q,t),\sigma^2,I_o,\lambda^*_1) = \oint_{\Gamma} g(\nabla I(C(q,t))) ds - \lambda^*_1 \int_{\Omega_0} (D(x,y) - T_x) dxdy + \lambda \left( E_p - T_p \right) \]  
(19)

where \( \lambda > 0 \) is another Lagrange multiplier. \( E_p \) is a convex quadratic function [15] with the global minimum at \( \Gamma(q) \). With the introduction of \( E_p \) and a large \( \lambda \), \( l_p(C(q,t),\sigma^2,I_o,\lambda^*_1) \) is convexified with the global minimum at target boundary \( \Gamma(q) \). We can take a derivative based approach to minimize Eq. (15) and the updating of \( \lambda^*_1 \) by letting \( \lambda^*_1 = \lambda^*_1 \). The problem is that the approach is a local method and often fails to locate global energy minimum.

The derivations of the proposed approach

Consider \( C(q,t) \) as a random function of time \( t \) and parameters \( \sigma^2, I_o, \) and \( \lambda^*_1 \) as random variables.

According to MFA approach, \( C(q,t), \sigma^2, I_o, \) and \( \lambda^*_1 \) can be regarded as all random functions of temperature \( T \) respectively denoted as \( C(q,t), \sigma^2(T), I_o(T), \) and \( \lambda^*_1(T) \). \( \sigma^2_c, I^*_o, \) and \( \lambda^*_1 \), which are obtained from \( C(q,t), T \), are also functions of \( T \) denoted as \( \sigma^2_c(T), I^*_o(T), \) and \( \lambda^*_1 \). The means of \( \sigma^2(T), I_o(T), \) and \( \lambda^*_1(T) \) can be computed according to Eq. (3).

Specifically, noting that in Eq. (20), \( \lambda > \lambda^*_1 \) and \( L_p \approx E_p \), we have

\[ I_o(T) = \frac{\int_{-\infty}^{\infty} \exp \left( -\frac{L_p}{T} \right) dT}{\int_{-\infty}^{\infty} \exp \left( -\frac{E_p}{T} \right) dT} \approx \frac{\int_{-\infty}^{0} \exp \left( -\frac{E_p}{T} \right) dT}{\int_{-\infty}^{0} \exp \left( -\frac{E_p}{T} \right) dT} = I^*_o(T) \]  
(21)

Since \( \sigma^2(T) > 0 \), \( \sigma^2(T) \) can be computed as
by 0. Thus we have

$\sigma^2(T)=\int_0^\infty \sigma^2 \exp\left(-\frac{L_\sigma}{T}\right) d\sigma^2 \approx \int_0^\infty \sigma^2 \exp\left(-\frac{E_u}{T}\right) d\sigma^2$

$=\frac{2}{\sqrt{T}} \int_0^\infty \exp\left(-\frac{\sigma^2}{T}\right) + \sigma^2(T)$ (22)

Since typically $T >> \sigma^2$, to facilitate computation, we can approximate $-\frac{\sigma^2}{\sqrt{T}}$ by 0. Thus we have

$\sigma^2(T)=\sigma^2(T) + \frac{T}{\sqrt{T}}$ (23)

Similarly we have $\lambda^2(T)=\lambda^2(T) + \frac{T}{\sqrt{T}}$ (24)

We use saddle point approximation to compute the mean of $C(q,t,T)$.

$Z_C \approx M \exp\left(\frac{-1}{T}L_\sigma(C(q,t,T),\sigma^2(T),I_\sigma(T),\lambda^2(T))\right)$, where

$\frac{\partial L_\sigma(C(q,t,T),\sigma^2(T),I_\sigma(T),\lambda^2(T))}{\partial C(q,t,T)} = 0$ (25)

with

$\frac{\partial C(q,t,T)}{\partial t} = [\lambda_i(T)][D(x,y,T)-T_i] + k g(\nabla|I|) - \nabla g \cdot \hat{N}$ (26)

where $D(x,y,T) = (I(x,y)-I_\sigma(T))^2 / 2\sigma^2(T)$

In practical applications, we relax Eq. (25) as

$\frac{\partial C(q,t,T)}{\partial t} = 0$ (27)

where $T_C$ is a given constant. To reduce computational complexity, instead of satisfying Eq. (25), contour $C(q,t,T)$ is deformed for a certain time interval $I > 0$ and we assume that Eq. (27) can be satisfied after the interval.

Comments: According to MFA theory [7], we can ignore the correlations among the mean fields of $C(q,t,T)$, $\sigma^2(T)$, $I_\sigma(T)$, $\lambda^2(T)$ and proceed to update each variable separately by holding all other parameters unchanged. Specifically, the solution includes two alternating procedures: i) deform $C(q,t,T)$ according to Eq. (26) with $\sigma^2(T) = \sigma^2(T)$, $I_\sigma(T) = I_\sigma(T)$, $\lambda^2(T) = \lambda^2(T)$ and ii) parameter updating of Eq. (21) to (23) with $\sigma^2(T)$, $I^C$, and $\lambda^C$ being obtained from $C(q,t,T)$. We track $C(q,t,T)$, $\sigma^2(T)$, $I_\sigma(T)$, $\lambda^2(T)$ as $T$ approaches zero. Typically, the method starts with an initial contour of size 5 by 5 inside target object and grows outward to the boundary. In Eq. (22) and Eq. (26), $\sigma^2(T)$ is large when temperature is high and $\sigma^2(T)$ gradually reduces when the temperature lowers as the contour grows outward. It is also interesting to note that according to Eq. (24) the magnitude of the balloon force term in Eq. (26) is decaying during the annealing process. This feature provides the method more robustness to noisy interiors while enhancing its performances in situations of gaps and blur boundaries. In general, the whole contour deformation process can be viewed as an annealing process, in which contour flows outward in a high temperature and then cools down and anneals near the target boundary.

Algorithm description

The algorithm can thus be derived as:

For an initial contour $C(q, t, T_i)$ with the size of 5 by 5, $t = 0$ and interior $\Omega_0(t)$, at temperature $T_i = T_{ini}$, $i = 1$, where $T_{ini}$ is the initial temperature, let $C(q, t, T_i) = C(q, t, T_{ini})$ do

i) Compute $\sigma^2(T)$, $I_\sigma(T)$, and $\lambda^2(T)$ from $C(q, t, T_i)$ using Eq. (17), Eq. (18), and Eq. (19).

ii) Update $\sigma^2(T)$, $I_\sigma(T)$, and $\lambda^2(T)$ using Eq. (23), Eq. (21), and Eq. (24).

iii) Let $T_{ini} = T_i \cdot decT$, where $0 < decT < 1$ is the updating factor.

iv) Evolve $C(q, t, T_i)$ for $l$ iterations ($l > 0$ is a constant) according to Eq. (26) using narrow band numerical scheme [3] with $C(q,0,T_i) = C(q, t, T_{ini})$ being the initial contour. Let $C(q, t, T_l) = C(q, t, T_l)$.

Stop when the maximum velocity of $C(q, t, T_l)$ is smaller than a threshold $V$, or a maximum iteration number $t_m$ has been reached. Otherwise go to Step i).

$V$ and $t_m$ are positive constants. To further increase the robustness of the algorithm to the setting of iteration number $t_m$, we compute the value of contour energy according to Eq. (7) and record the lowest energy contour. The output contour is then the contour with lowest contour energy.

It should be noted that we can also set the stopping criteria as when the temperature $T_i$ drops to zero.
However, this stopping criterion is too sensitive to the settings of $T_{int}$ and $decT$ thus is not used.

5. Illustrative Examples

In this section, we will show that different target boundaries in a single image can be modeled as the constrained global minima with different settings of $T_p$. We then illustrate the processes of locating these constrained global energy minima using the proposed method with a comparison to the results using Eq. (15) of CODCM [1] assuming that $\sigma^2$ and $I_0$ of the target objects are known.

The examples shown in Fig. 6.1(a) is a 165 by 165 image designed with three overlapping circles $MC_1$, $MC_2$, $MC_3$ of size $r_1 = 40$ pixels ($MC_1$), $r_2 = 55$ pixels ($MC_2$), $r_3 = 70$ pixels ($MC_3$) at different center locations, where $r_i$, $i = 1, 2, 3$ is the radius of circle $MC_i$. The interior brightness of circles $MC_1$ and $MC_3$ is radially decreasing from the centers to their perimeters in a straight line fashion from 40 to 150. The values of $\sigma^2$, $I_0$, and $D_{min}(x, y)$ inside $MC_1$, $MC_2$, $MC_3$, and the respective contour energy $E_c$ (computed according to Eq. (7)) are listed in the Table 6.1, where $D_{min}(x, y)$ is the minimum of $D(x, y)$,

<table>
<thead>
<tr>
<th>$D_{min}(x, y)$</th>
<th>$\sigma^2$</th>
<th>$I_0$</th>
<th>$E_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MC_1$</td>
<td>0.016</td>
<td>127.7</td>
<td>223</td>
</tr>
<tr>
<td>$MC_2$</td>
<td>0.289</td>
<td>2359</td>
<td>178</td>
</tr>
<tr>
<td>$MC_3$</td>
<td>0.174</td>
<td>3682</td>
<td>142</td>
</tr>
</tbody>
</table>

Table 6.1 The parameters and contour energy for $MC_1$, $MC_2$, and $MC_3$

It is easy to see that when $T_p < 0.016$, $0.016 < T_p < 0.174$, and $0.174 < T_p < 0.289$, $MC_1$, $MC_3$, and $MC_2$ are the constrained global energy minimum, respectively. Thus $MC_1$, $MC_3$, $MC_2$ can all be modeled as the constrained global energy minimum by setting different values of $T_p$, respectively $T_p = 0.01$ for $MC_1$, $T_p = 0.1$ for $MC_3$, $T_p = 0.25$ for $MC_2$. The extraction processes of $MC_1$, $MC_2$, $MC_3$ are shown in Fig. 6.1b to 6.1d, 6.2a to 6.2d, and 6.3a to 6.3d. With the same initial contour position indicated as the black dot in Fig. 6.1a, the method successfully locates all the constrained global energy minimum contours. The results of [1] using the same initial contour in Fig. 6.1a and Eq. (15) by assuming that $I_0$ and $\sigma^2$ of $MC_1$, $MC_2$, and $MC_3$ are known (as listed in Table 6.1) are shown in Fig. 6.4b to 4c, 5a to 5c, and 6a to 6c. Though Eq. (15) successfully extracts $MC_1$, it fails to extract $MC_2$ and $MC_3$.

6. Applications

Our experimental evaluations and comparisons can be divided into three separate items and we separately discuss each category as follows:

i) Contour evaluations are performed on a set of challenging contour extraction problems including ultrasound pig heart images having noisy contour interiors, sharp contour segment protrusions, and gaps as shown in Fig. 8.1, MRI knee images having thick and very blur contour segment and contour-within-contour segment as shown in Fig. 8.2, and MRI brain images having contours with complex shape, inhomogeneous interiors, and blur segments as shown in Figs. 8.3, 8.4, and 8.5. Resulting contours are shown on the right-hand side of their originals in Figs. 8.1, 8.2, 8.4. Notice that for the MRI brain images, we perform three separate extractions of external boundary of intracranial (Fig. 8.3), cerebral boundary (Fig. 8.4), and sulci boundary (Fig. 8.5). Furthermore, Fig. 8.3 shows a sequence of contour progression images. All these contours are considered very good results.

ii) A performance comparison between the proposed method and CODCM [16] is made on four images shown in Figs. 8.6, and 8.7 including visual blood cell image, MRI knee image, and two MRI brain images. In Fig. 8.6a, the cells have rather large gaps. In Fig. 8.7a and 8.7d, there are very inhomogeneous interiors inside the external boundaries of intracranial. In Fig. 8.7g, there are blur boundary segment and a rather inhomogeneous interior. As we see from the results shown in Fig. 8.6b, 8.6c, 8.7b, 8.7c, 8.7e, 8.7f, 8.7h, and 8.7i, comparing to the result of [16], substantial improvements can be seen.

iii) The proposed method is also compared to other conventional deformable contour methods; the first two methods are geodesic snake [4], and area & length active contour [10] using $h(x, y) = \frac{1}{1 + \sqrt{G * I^2}}$ as the edge detection function and the third method is T-snake [11]. We select two zoomed images of Fig. 8.8a a stomach CT image with additive noise of Gaussian noise (variance 3000), and Fig. 8.9a a midline sagittal MRI brain image. Similar to all other three methods, a
Gaussian filter $N(0, 1)$ is applied to both images as a preprocessing operation. No a priori information of object shape or brightness distribution is assumed. To provide an objective comparison, two sets of three dark dots in Fig. 8.8a and Fig. 8.9a are used as initial candidate locations for all four methods including ours. Each method using an initial candidate location provides a resulting contour. The best contour (best of the three resulting contours for Fig. 8.8a and Fig. 8.9a) of each method from all initial candidate locations is selected for comparison. Comparing these resulting contours, our proposed method has the best contours.

7. Conclusions

In this paper, a constrained global optimization formulation has been proposed for boundary extraction problems. The effectiveness of the approach in locating constrained global energy minima is evaluated in a synthetic test image, where constrained global energy minima are known. The performance of the method is demonstrated on very challenging segmentation applications and is compared to those of other deformable contour methods and substantial improvements are reported. The method is computationally efficient usually taking 10 seconds to 2 minutes on workstation Ultra Sun Blade 100 for most applications.

Reference


Fig. 5.1b to 5.1d, Fig. 5.2a to 5.2d, and Fig. 5.3a to 5.3d respectively illustrate the contour evolution processes of extracting $MC_1$, $MC_2$, and $MC_3$ using the proposed method. Fig. 5.4a to 5.4c, Fig. 5.5a to 5.5c, and Fig. 5.6a to 5.6c respectively illustrate the contour evolution processes of extracting $MC_1$, $MC_2$, and $MC_3$ using Eq. (17).
Fig. 6.1 Ultrasound pig heart original images in the left column and their corresponding results in the right column.

Fig. 6.2 MRI knee original images on the left column and their corresponding results on the right column.

Fig. 6.3a Original MRI brain image (white dot indicates the position of the initial contour). 6.3b to 6.3d The boundary extraction process of the exterior boundary of intracranial.

Fig. 6.4 MRI brain image two and the result.

Fig. 6.5 MRI brain image three and the result.

Fig. 6.6 Comparison results with CODCM [16]. 6.6b is the segmentation result of the cell using the proposed method. 6.6c is the segmentation result of the cell using CODCM [16].
Fig. 6.7 Comparison results with CODCM [16]. 6.7b, 6.7e and 6.7h are segmentation results using the proposed method. 6.7c, 6.7f, and 6.7i are segmentation results using CODCM [16].

Fig. 8.8a, 8.9a Original images of the comparison, 8.8b, 8.8b The results of T-snake, 8.8c, 8.9c The results of geodesic snake, 8.8d, 8.9d The results of Area-length snake, 8.8e, 8.9e The results of proposed method.